Lecture:

= Apply derivatives to finding integrals

Section 4.9. Antiderivatives.

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The Family of Antiderivatives Associated With a Function f

Antiderivative = opposite of derivative

Definition. F is an antiderivative of f if its derivative is equal to f, i.e.,

$$F'(x) = f(x).$$

Example. The function $f(x) = 3x^2$ is the derivative of another function $F(x)$, called the antiderivative of $f(x)$. Find (or guess at) $F(x)$.

$$F'(x) = f(x) \Rightarrow F'(x) = 3x^2$$

$$\Rightarrow F(x) = x^3$$

check: $F'(x) = \frac{d}{dx}(x^3) = 3x^2 = f(x)$ ✓

$$\left.\begin{array}{l} x^3 \xleftarrow{\text{differentiation}} 3x^2 \\ x^3 \xrightarrow[\text{antidifferentiation}]{} 3x^2 \end{array}\right\} \quad \underset{\text{antiderivative}}{x^3} \longleftrightarrow \underset{\text{derivative}}{3x^2}$$

Actually, other $F(x)$'s that will also work are:

$$F(x) = x^3 + 1 \quad ; \quad F'(x) = \frac{d}{dx}(x^3+1) = 3x^2 = f(x) \checkmark$$

$$F(x) = x^3 - 55 \quad ; \quad F'(x) = \frac{d}{dx}(x^3-55) = 3x^2 = f(x) \checkmark$$

$$F(x) = x^3 + \frac{3}{4} \quad ; \quad F'(x) = \frac{d}{dx}\left(x^3+\frac{3}{4}\right) = 3x^2 = f(x) \checkmark$$

All of these $F(x)$'s are antiderivatives of $f(x) = 3x^2$

So, we say that the <u>general antiderivative</u> or <u>family of antiderivatives</u> of $f(x) = 3x^2$ is

$$F(x) = x^3 + C \quad , \quad C = \text{any constant}$$

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<u>check</u>: $F'(x) = \frac{d}{dx}(x^3 + C) = 3x^2 + 0 = 3x^2 = f(x)$ ✓

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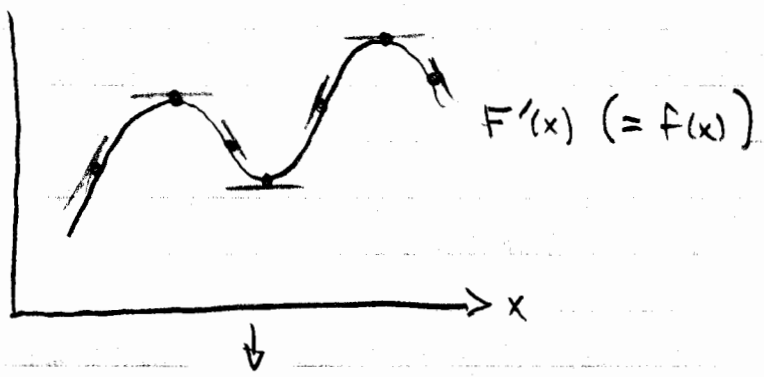
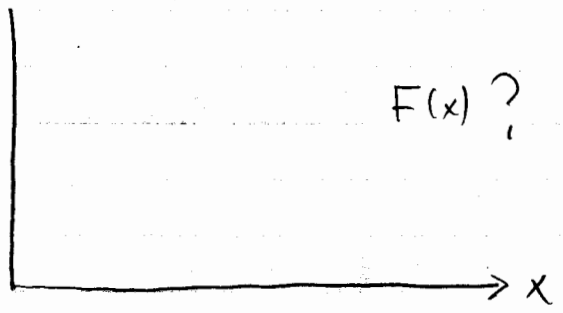
Visualizing Antiderivatives Using the Slopes of Tangent Lines (SEE Section 2.10, pp. 177-178, as Well as Section 4.9, p. 332)

Recall that

$$\underbrace{F'(a) = \text{slope of tangent line to } y = F(x) \text{ at } x = a}_{\text{derivative} \;=\; \text{slope}}$$

We will first use this idea in an attempt to sketch $F(x)$ given a sketch of $F'(x)$ ($= f(x)$, where $F(x)$ is the antiderivative of $f(x)$):



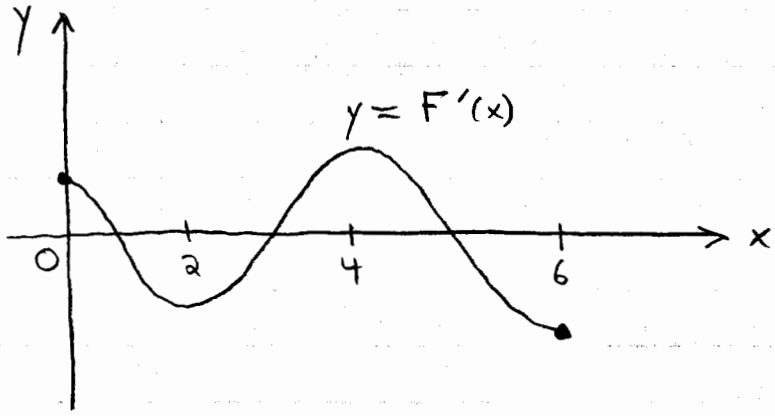


<u>Example</u>. (Exercise 2, p. 178, Sect. 2.10.)

The text uses f' instead

The graph of the derivative F' of a function f is shown:

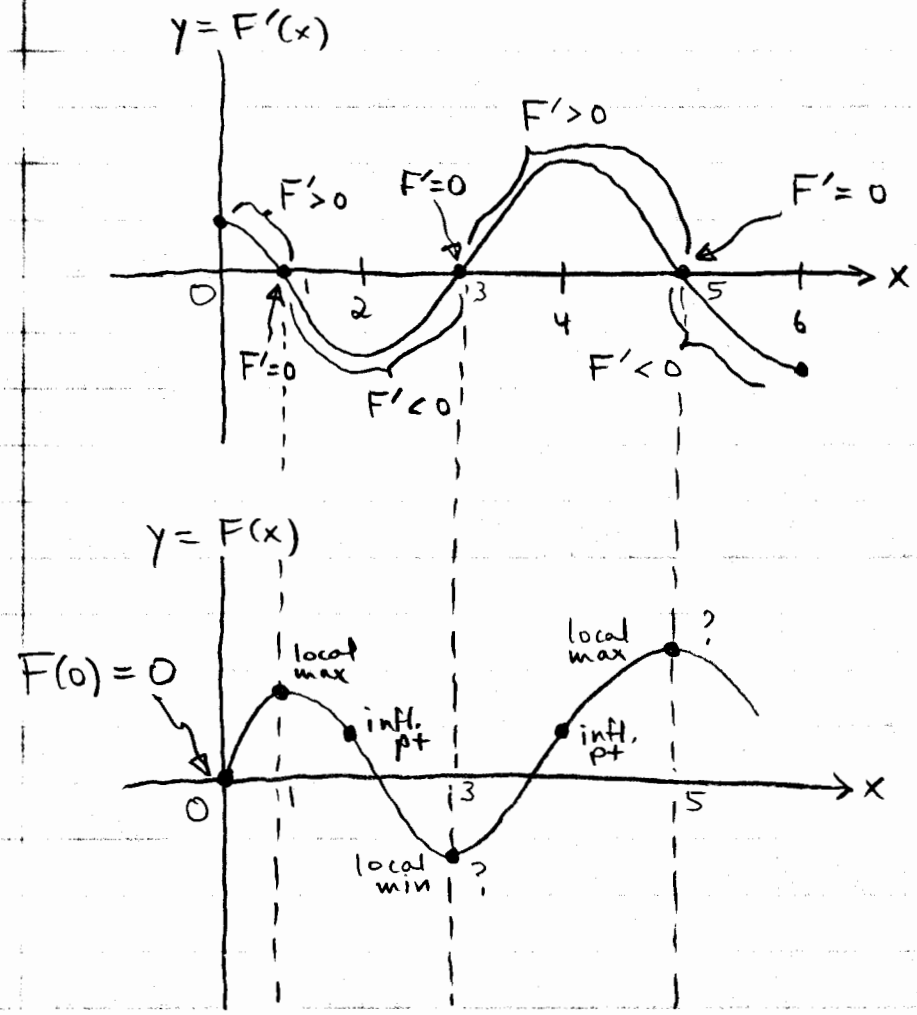
The text uses f instead



(a) On what intervals is F increasing or decreasing?

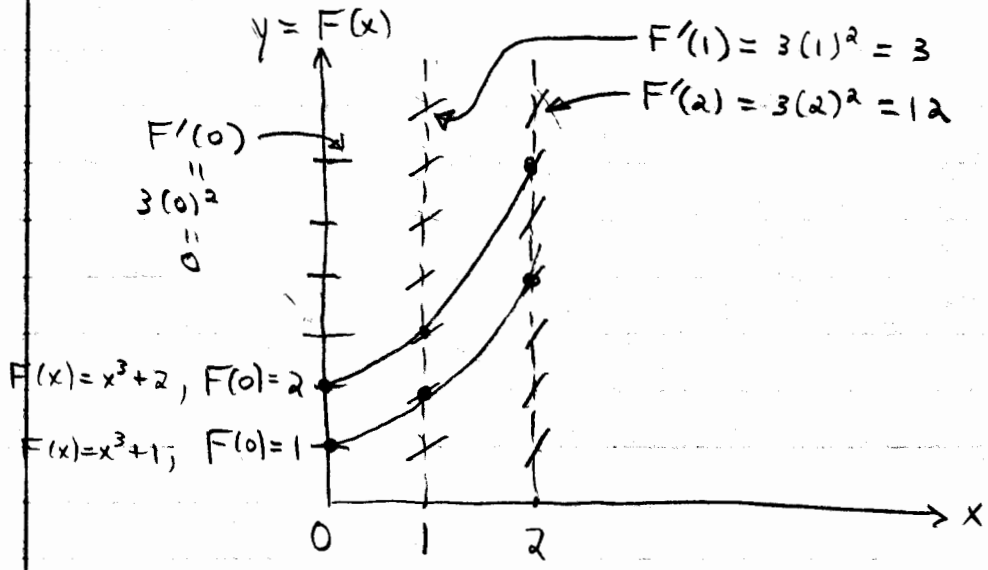
(b) At what values of x does F have a local maximum or minimum?

(c) If it is known that $F(0) = 0$, sketch a possible graph of F.



F'	F
> 0	increasing
< 0	decreasing
$= 0$	local max ∧ or local min ∨ or "seat"
(increasing	concave up)
(decreasing	concave down)
local max or min	inflection point

We will now sketch $F(x)$ by connecting a whole bunch of short tangent lines that come from $F'(x)$ $(= f(x))$:



E.g., $F(x) = x^3 + C$, $F'(x) = 3x^2$

Called a DIRECTION FIELD for $f(x) = 3x^2$

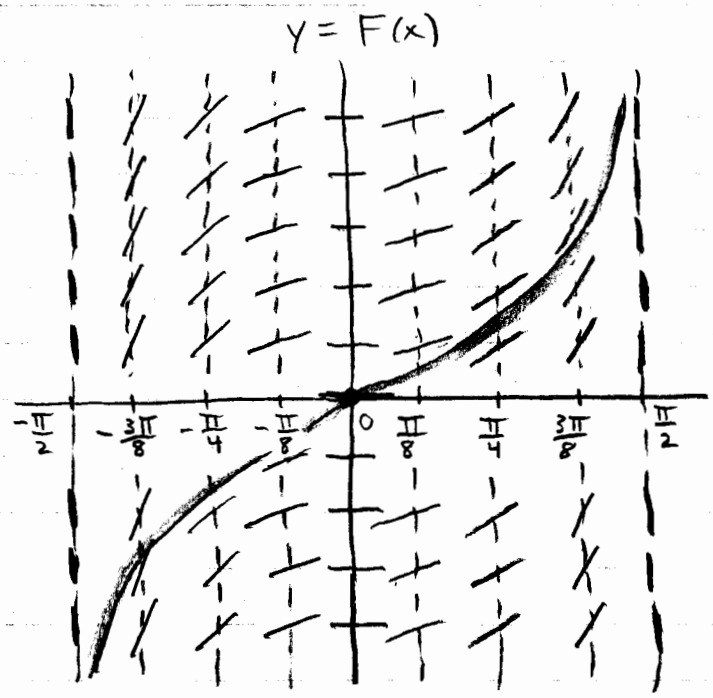
<u>Example</u>. (Exercise 32, p. 334, Sect. 4.9.)

Use a direction field to graph the antiderivative of

$$f(x) = x \tan x \qquad -\pi/2 < x < \pi/2$$

that satisfies

$$F(0) = 0.$$



$f(x) = x \tan x =$ slope of tangent lines at x

x	$f(x) = x\tan x$ = slope of tangent lines at x
0	0
$\pm \pi/8$	0.16
$\pm \pi/4$	0.78
$\pm 3\pi/8$	2.84
$\pm \pi/2$	∞

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= Will make Tables of antiderivatives

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Constructing Antiderivatives
(SEE Section 5.3, pp. 371-372, as Well as Section 4.9, pp. 329-331)

The

ANTIDERIVATIVE

of a function f is also called the

(INDEFINITE) INTEGRAL

of a function f.

The indefinite integral of a function f is denoted by

$$\int f(x)\,dx = \underbrace{F(x) + C}$$

General antiderivative (or family of antiderivatives) of $f(x)$ where

$$\frac{d}{dx}\left(F(x) + C\right) = F'(x) + 0$$
$$= F'(x)$$
$$= f(x)$$

So we have

DIFFERENTIATION, $\frac{d}{dx}$,

and

INTEGRATION, $\int dx$,

two reverse processes :

DIFFERENTIATION

$$\frac{d}{dx}\left(F(x)+C\right) = F'(x) = f(x)$$

$$\begin{tikzcd}
F(x)+C \arrow[rr, bend left] & & f(x) \arrow[ll, bend left] \\
\text{ANTIDERIVATIVE} & & \text{DERIVATIVE}
\end{tikzcd}$$

$$\int f(x)\,dx = F(x) + C$$

INTEGRATION

Examples.

1. Antiderivative of $f(x) = 0$?
Ask: What function has derivative $= 0$?
Answer: Any constant function $F(x) = C$.
So

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$$\int 0 \, dx = C.$$

2. Antiderivative of $f(x) = 3$?
11/20/01 Ask: What function has derivative $= 3$?
Tues. Answer: The linear function $F(x) = 3x$.
So

$$\int 3 \, dx = 3x \underbrace{+ C}_{\text{"Tag on" a } C.}$$

3. Antiderivative of $f(x) = x$?
Ask: What function has derivative $= x$?
Answer: Try $F(x) = x^2$. Then

$$F'(x) = 2x \underset{\substack{\text{almost} \\ \text{but not} \\ \text{quite}}}{\neq} x = f(x)$$

Try $F(x) = \frac{1}{2}x^2$. Then

$$F'(x) = x = f(x) \checkmark$$

So

$$\int x\,dx = \frac{1}{2}x^2 + \underbrace{C}_{\text{"Tag on" a } C.}$$

4. Antiderivative of $f(x) = x^2$?
Ask: What function has derivative $= x^2$?
Answer: Try $F(x) = x^3$. Then

$$F'(x) = 3x^2 \underset{\text{almost}}{\neq} x^2 = f(x)$$

Try $F(x) = \frac{1}{3}x^3$. Then

$$F'(x) = x^2 = f(x) \checkmark$$

So

$$\int x^3\,dx = \frac{1}{3}x^3 + \underbrace{C}_{\text{"Tag on" a } C.}$$

In general:

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1}$$

OR $\frac{1}{n+1} \cdot x^{n+1}$

Otherwise, can get

$$\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} = \text{undefined!}$$

More Examples.

1. Antiderivative of $f(x) = e^x$?
What function has derivative $= e^x$?
Recall: $\frac{d}{dx}(e^x) = e^x$.

So

$$\boxed{\int \underbrace{e^x}\, dx = \underbrace{e^x + C}}$$

Derivative of this must equal this

2. Antiderivatives of $f(x) = \sin x$ and $f(x) = \cos x$?
What function has derivative $= \sin x$? $= \cos x$?
Recall: $\frac{d}{dx}(\cos x) = -\sin x \Rightarrow$

$$\frac{d}{dx}(-\cos x) = -\frac{d}{dx}(\cos x) = -(-\sin x) = \sin x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

So

$$\boxed{\begin{aligned} \int \sin x\, dx &= -\cos x + C \\ \int \cos x\, dx &= \sin x + C \end{aligned}}$$

3. Antiderivative of $f(x) = x^{-1} = \frac{1}{x}$?

NOTE: Cannot use formula

$$\int x^n \, dx = \frac{x^n}{n+1} + C .$$

What function has derivative $= \frac{1}{x}$?
Actually, there are two!

$x > 0$: $F(x) = \ln x$ — positive

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$x < 0$: $F(x) = \ln(-x)$ — positive

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot \frac{d}{dx}(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

CHAIN RULE

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x)$$

Can put these two $F(x)$'s together into one function

$$F(x) = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

So

$$\boxed{\int \frac{1}{x} \, dx = \ln|x| + C}$$

Properties of Antiderivatives (or Indefinite Integrals)

1. $$\boxed{\int \left(f(x) \pm g(x) \right) dx = \int f(x)\, dx \pm \int g(x)\, dx}$$

E.g., $$\int \left(x^2 + \frac{1}{x} \right) dx = \int x^2 dx + \int \frac{1}{x}\, dx$$

$$= \frac{x^3}{3} + \ln|x| \underbrace{+ C}$$

Just "tag on" <u>one</u> C

2. $$\boxed{\int c f(x)\, dx = c \cdot \int f(x)\, dx}$$

E.g., $$\int 3e^x\, dx = 3 \cdot \int e^x\, dx$$

$$= 3e^x \underbrace{+ C}$$

"Tag on" C last

<u>Examples</u>. Find the most general antiderivative, $F(x)+C$, of the function, $f(x)$.

1. (Exercise 2, p. 334)

$f(x) = 1 - x^3 + 12x^5$

$$\int f(x)\,dx = \int (1 - x^3 + 12x^5)\,dx$$

$$= \int x^0\,dx - \int x^3\,dx + 12\int x^5\,dx$$

$$= \frac{x^{0+1}}{0+1} - \frac{x^{3+1}}{3+1} + 12\cdot\frac{x^{5+1}}{5+1} + C$$

$$= \frac{x^1}{1} - \frac{x^4}{4} + \cancel{12}^{\,2}\cdot\frac{x^6}{\cancel{6}} + C$$

$$= \boxed{x - \frac{x^4}{4} + 2x^6 + C}$$

Check: $\frac{d}{dx}\left(x - \overset{=\frac{1}{4}x^4}{\frac{x^4}{4}} + 2x^6 + C\right)$

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Tues.

$$= \frac{d}{dx}(x) - \frac{1}{4}\frac{d}{dx}(x^4) + 2\frac{d}{dx}(x^6) + \frac{d}{dx}(C)$$

$$= 1 - \frac{1}{\cancel{4}}(\cancel{4}x^3) + 2(6x^5) + 0$$

$$= 1 - x^3 + 12x^5 \checkmark$$

3. (Exercise 10, p. 334.)

$f(x) = 3e^x + 7\sec^2 x$

$$\int f(x)\,dx = \int (3e^x + 7\sec^2 x)\,dx$$

$$= 3\cdot\int e^x\,dx + 7\cdot\int \sec^2 x\,dx$$

$$= \boxed{3e^x + 7\tan x + C}$$

Recall: $\frac{d}{dx}(\tan x) = \sec^2 x$

4. (Exercise 12, p. 334.)

$$f(x) = \frac{x^2+x+1}{x} \underset{\text{HINT}}{=} \frac{x^2}{x} + \frac{x}{x} + \frac{1}{x} = x+1+\frac{1}{x}$$

$$\int f(x)\,dx = \int \left(x+1+\frac{1}{x}\right) dx$$

$$= \int x\,dx + \int 1\,dx + \int \frac{1}{x}\,dx$$

$$= \boxed{\frac{x^2}{2} + x + \ln|x| + C}$$

<u>Example</u>. (Exercise 14, p. 334.)

Find antiderivative F of f that satisfies given condition.

$$f(x) = 4 - 3(1+x^2)^{-1} = 4 - 3 \cdot \frac{1}{1+x^2},$$

$$\underbrace{F(0) = 4}$$

Called "given condition" or "initial condition." We will use this to determine a specific value for C.

$$\int f(x)\,dx = \int \left(4 - 3 \cdot \frac{1}{1+x^2}\right) dx$$

$$= 4 \cdot \int 1\,dx - 3 \cdot \int \frac{1}{1+x^2}\,dx$$

$$= 4x - 3\tan^{-1} x + C = F(x)$$

Recall: $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

Apply $F(0) = 4$ to $F(x) = 4x - 3\tan^{-1} x + C$ to determine C:

$$4 = F(0) = 4(0) - 3\tan^{-1}(0) + C \Longrightarrow$$

$$4 = 4(0) - 3\underbrace{\tan^{-1}(0)}_{=0 \text{ by calculator}} + C \Longrightarrow$$

$$4 = 0 - 3(0) + C \Longrightarrow$$

$$4 = C \Longrightarrow$$

$$C = 4$$

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$$\therefore \boxed{F(x) = 4x - 3\tan^{-1}x + 4}$$