The Family of Antiderivatives Associated With a Function $f$

Antiderivative = opposite of derivative

Definition: $F$ is an antiderivative of $f$ if its derivative is equal to $f$, i.e.,

$$ F'(x) = f(x). $$

Example. The function $f(x) = 3x^2$ is the derivative of another function $F(x)$, called the antiderivative of $f(x)$. Find (or guess at) $F(x)$,

$$ F'(x) = f(x) \Rightarrow F'(x) = 3x^2 $$

$$ \Rightarrow F(x) = x^3 $$

check: $F''(x) = \frac{d}{dx}(x^3) = 3x^2 = f(x) \checkmark$
Actually, other $F(x)$'s that will also work are:

$F(x) = x^3 + 1$ : $F'(x) = \frac{d}{dx}(x^3 + 1) = 3x^2 = f(x)$ ✓

$F(x) = x^3 - 55$ : $F'(x) = \frac{d}{dx}(x^3 - 55) = 3x^2 = f(x)$ ✓

$F(x) = x^3 + \frac{3}{4}$ : $F'(x) = \frac{d}{dx}(x^3 + \frac{3}{4}) = 3x^2 = f(x)$ ✓

All of these $F(x)$'s are antiderivatives of $f(x) = 3x^2$.

So, we say that the **general antiderivative** or **family of antiderivatives** of $f(x) = 3x^2$ is

$$F(x) = x^3 + C \quad C = \text{any constant}$$
- 287(12) -

\[ F'(x) = \frac{d}{dx}(x^3 + C) = 3x^2 + 0 = 3x^2 = f(x) \checkmark \]
Visualizing Antiderivatives Using
the Slopes of Tangent Lines
(SEE Section 2.10, pp. 197-198, as
Well as Section 4.9, p. 332)

Recall that

\[ F'(x) = \text{slope of tangent line to } y = F(x) \text{ at } x = a \]

\text{derivative } = \text{slope}

We will first use this idea in an attempt to
sketch \( F(x) \) given a sketch of \( F'(x) = f(x) \), where
\( F(x) \) is the antiderivative of \( f(x) \):
Example. (Exercise 2, p. 174, Sect. 2.10)

The graph of the derivative $F'$ of a function $F$ is shown:

(a) On what intervals is $F$ increasing or decreasing?

(b) At what values of $x$ does $F$ have a local maximum or minimum?

(c) If it is known that $F(0) = 0$, sketch a possible graph of $F$. 

\[ y = F'(x) \]
\[ y = F'(x) \]

- **F'(x)**
  - **F' > 0**
  - **F' < 0**
  - **F' = 0**

- **F(x)**
  - **F(0) = 0**
  - **local max**
  - **local min**
  - **infl. pt.**

**Table: F' vs. F**

<table>
<thead>
<tr>
<th>F'</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 0</td>
<td>increasing</td>
</tr>
<tr>
<td>= 0</td>
<td>local max ( \forall ) or local min ( \exists ) or &quot;plateau&quot;</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>decreasing</td>
</tr>
<tr>
<td></td>
<td>concave up</td>
</tr>
<tr>
<td></td>
<td>concave down</td>
</tr>
<tr>
<td></td>
<td>inflection point</td>
</tr>
</tbody>
</table>

\[ \boxed{284(15)} \]
We will now sketch $F(x)$ by connecting a whole bunch of short tangent lines that come from $F'(x) (= f(x))$.

\[ F(x) = x^3 + C, \quad F'(x) = 3x^2 \]

Collide a DIRECTION FIELD for $f(x) = 3x^2$. 
Example. (Exercise 52, p. 334, Sect. 4.9.)

Use a direction field to graph the antiderivative of

\[ f(x) = x \tan x, \quad -\pi/2 < x < \pi/2 \]

that satisfies

\[ F(0) = 0. \]

\[ f(x) = x \tan x = \text{slope of tangent lines at } x \]
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x \tan x$ = slope of tangent lines at $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\pm \pi/8$</td>
<td>$0.16$</td>
</tr>
<tr>
<td>$\pm \pi/4$</td>
<td>$0.78$</td>
</tr>
<tr>
<td>$\pm 3\pi/8$</td>
<td>$2.84$</td>
</tr>
<tr>
<td>$\pm \pi/2$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
Constructing Antiderivatives

(See Section 5.3, pp. 371-372,
as well as Section 4.9, pp. 329-331)

The

**ANTIDERIVATIVE**

of a function \( f \) is also called the

(INDEFINITE) INTEGRAL

of a function \( f \).

The indefinite integral of a function \( f \)
is denoted by

\[
\int f(x) \, dx = F(x) + C
\]

General antiderivative (or family of antiderivatives) of \( f(x) \) where

\[
\frac{d}{dx} (F(x) + C) = F'(x) + 0 = F'(x) = f(x)
\]
So we have

Differentialiation, \( \frac{d}{dx} \)

and

Integration, \( \int dx \)

two reverse processes:

**Differentialiation**

\[ \frac{d}{dx}(F(x) + C) = F'(x) = f(x) \]

\[ F(x) + C \quad \text{Antiderivative} \quad \uparrow \quad f(x) \quad \text{Derivative} \]

\[ \int f(x) \, dx = F(x) + C \]

**Integration**
Examples:

1. Antiderivative of \( f(x) = 0 \)?
   Ask: What function has derivative \( 0 \)?
   Answer: Any constant function \( F(x) = C \).
   So
   \[
   \int 0 \, dx = C.
   \]

2. Antiderivative of \( f(x) = 3 \)?
   Ask: What function has derivative \( 3 \)?
   Answer: The linear function \( F(x) = 3x \).
   So
   \[
   \int 3 \, dx = 3x + C \quad \text{"Tag on" a } C.
   \]

3. Antiderivative of \( f(x) = x \)?
   Ask: What function has derivative \( x \)?
   Answer: Try \( F(x) = x^2 \).
   Then
   \[
   F'(x) = 2x \neq x = f(x)
   \]
   Try \( F(x) = \frac{1}{2} x^2 \).
   Then
   \[
   F'(x) = x = f(x) \checkmark
   \]
\[ \int x \, dx = \frac{1}{2} x^2 + C \]

"Tag on" a \( C \).

4. Antiderivative of \( f(x) = x^2 \)?

Ask: What function has derivative \( x^2 \)?

Answer: Try \( F(x) = x^3 \). Then

\[ F'(x) = 3x^2 \neq x^2 = f(x) \]

Try \( F(x) = \frac{1}{3} x^3 \). Then

\[ F'(x) = x^2 = f(x) \checkmark \]

So

\[ \int x^2 \, dx = \frac{1}{3} x^3 + C \]

"Tag on" a \( C \).
In general:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

or \( \frac{1}{n+1}, x^{n+1} \)

Otherwise, we can get:

$$\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{0}{0} = \text{undefined}!$$
1. Antiderivative of \( f(x) = e^x \)?
   What function has derivative \( e^x \)?
   Recall: \( \frac{d}{dx}(e^x) = e^x \).

So
\[
\int e^x \, dx = e^x + C
\]
(derivative of this must equal this)

2. Antiderivatives of \( f(x) = \sin x \) and \( f(x) = \cos x \)?
   What function has derivative \( \sin x \)? \( = \cos x \)?
   Recall: \( \frac{d}{dx}(\cos x) = -\sin x \Rightarrow \)
   \( \frac{d}{dx}(-\cos x) = -\frac{d}{dx}(\cos x) = -(\sin x) = \sin x \)
   \( \frac{d}{dx}(\sin x) = \cos x \)

So
\[
\int \sin x \, dx = -\cos x + C
\]
\[
\int \cos x \, dx = \sin x + C
\]
3. Antiderivative of \( f(x) = x^{-1} = \frac{1}{x} \)?

**NOTE:** Cannot use formula
\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.
\]

What function has derivative \( \frac{1}{x} \)?

Actually, there are two!

\[ x > 0: \quad F(x) = \ln(x) \quad \text{positive} \]
\[
\frac{d}{dx}(F(x)) = \frac{d}{dx}(\ln x) = \frac{1}{x}
\]

\[ x < 0: \quad F(x) = \ln(-x) \quad \text{negative} \]
\[
\frac{d}{dx}(F(x)) = \frac{d}{dx}(\ln(-x)) = \frac{1}{x} \cdot \frac{d}{dx}(-x) = \frac{x}{x} \cdot (1) = \frac{1}{x}
\]

**Chain Rule**
\[
\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x)
\]

Can put these two \( F(x) \)’s together into one function

\[
F(x) = \ln|x| = \begin{cases} \ln x & , \quad x > 0 \\ \ln(-x) & , \quad x < 0 \end{cases}
\]

So,
\[
\int \frac{1}{x} \, dx = \ln|x| + C
\]
Properties of Antiderivatives
(or Indefinite Integrals)

1. \[ \int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \]
   
   Example:
   \[ \int (x^2 + \frac{1}{x}) \, dx = \int x^2 \, dx + \int \frac{1}{x} \, dx \]
   
   \[ = \frac{x^3}{3} + \ln |x| + C \]

   Just "tag on" one \( C \)

2. \[ \int c f(x) \, dx = c \int f(x) \, dx \]
   
   Example:
   \[ \int 3e^x \, dx = 3 \int e^x \, dx \]
   
   \[ = 3e^x + C \]

   "Tag on" \( C \) lost
Examples. Find the most general antiderivative $F(x) + C$ of the function $f(x)$.

1. (Exercise 2, p. 334)

$f(x) = 1 - x^3 + 12x^5$

$$
\int f(x)\,dx = \int (1 - x^3 + 12x^5)\,dx \\
= \int x^0\,dx - \int x^3\,dx + 12\int x^5\,dx \\
= \frac{x^{0+1}}{0+1} - \frac{x^{3+1}}{3+1} + 12\cdot\frac{x^{5+1}}{5+1} + C \\
= \frac{x^1}{1} - \frac{x^4}{4} + 12\cdot\frac{x^6}{6} + C \\
= x - \frac{x^4}{4} + 2x^6 + C
$$

Check: \( \frac{d}{dx} \left( x - \frac{x^4}{4} + 2x^6 + C \right) \)

\[\begin{align*}
11/20/01 & \quad \text{Tues,} \\
= \frac{d}{dx}(x) - \frac{1}{4}\frac{d}{dx}(x^4) + 2\frac{d}{dx}(x^6) + \frac{d}{dx}(C) \\
= 1 - \frac{1}{4}(4x^3) + 2(6x^5) + 0 \\
= 1 - x^3 + 12x^5 \quad \checkmark
\]
3. (Exercise 10, p. 334.)

\[ f(x) = 3e^x + 7\sec^2 x \]

\[
\int f(x) \, dx = \int (3e^x + 7\sec^2 x) \, dx \\
= 3\int e^x \, dx + 7\int \sec^2 x \, dx \\
= 3e^x + 7\tan x + C
\]

Recall: \( \frac{d}{dx}(\tan x) = \sec^2 x \)
4. (Exercise 12, p. 334.)

\[ f(x) = \frac{x^2 + x + 1}{x} \]  

**HINT**

\[ \int f(x) \, dx = \int \left( x + 1 + \frac{1}{x} \right) \, dx \]

\[ = \int x \, dx + \int 1 \, dx + \int \frac{1}{x} \, dx \]

\[ = \frac{x^2}{2} + x + \ln|x| + C \]
Example: (Exercise 14, p. 334.)

Find antiderivative $F$ of $f$ that satisfies given condition.

$F(x) = 4 - 3 \left(1 + x^2\right)^{-1} = 4 - 3 \cdot \frac{1}{1 + x^2}$

$F(0) = 4$

Called "given condition" or "initial condition." We will use this to determine a specific value for $C$.

\[
\int f(x) \, dx = \int \left(4 - 3 \cdot \frac{1}{1 + x^2}\right) \, dx
\]

\[
= 4 \int dx - 3 \int \frac{1}{1 + x^2} \, dx
\]

\[
= 4x - 3 \tan^{-1} x + C = F(x)
\]

Recall: \( \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2} \)

Apply $F(0) = 4$ to $F(x) = 4x - 3 \tan^{-1} x + C$

to determine $C$:
\(- 2.87 \ (31) -

\begin{align*}
4 &= F(0) = 4(0) - 3\tan^{-1}(0) + C \\
4 &= 4(0) - 3\tan^{-1}(0) + C \\
&= 0 \text{ by calculator} \\
4 &= 0 - 3(0) + C \\
&= C \\
C &= 4
\end{align*}

\[
F(x) = 4x - 3\tan^{-1}x + 4
\]