Lecture

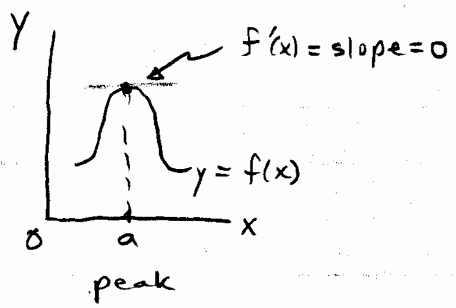
$\boxed{\text{CHAPTER 4}}$ Applications of Differentiation.

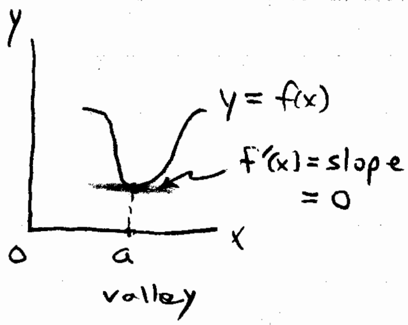
4/6/01 F Lecture

We will first apply IMPLICIT DIFFERENTIATION to find rates of change of processes such as the rate of increase of volume of an expanding balloon.

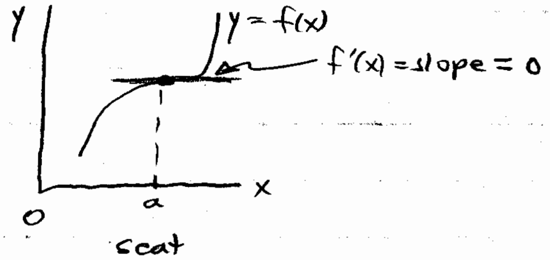
Then we will use the FIRST AND SECOND DERIVATIVES of a function to maximize or minimize quantities such as the cost of making a product. Involved here is the following idea:

$f'(a) = 0 \Rightarrow$ At $x = a$ we have a "peak" (max) "valley" (min) or "seat" (neither) in the graph of $y = f(x)$





4/6/01 F Lecture



We might be interested in the following rates of change:

4/6/01
F
Lecture

$\frac{dr}{dt}$ = rate of increase of radius per unit increase in time

$\frac{dA}{dt}$ = rate of increase of area per unit increase in time

$\frac{dA}{dr}$ = rate of increase of area per unit increase in radius

related rates involved here

$\frac{dC}{dt}$ = rate of circumference per unit increase in radius

etc.

In the rate of change problems we will consider, will mainly want to find

rate of change of a quantity Q per unit time

by knowing

time rates of change of quantities, say, r and s, that are related to Q

Lecture

Section 4.1. Related Rates.

4/6/01 F Lecture

Recall: The derivative $\frac{dy}{dx}$ gives instantaneous rate of change of $y = f(x)$ with respect to x, where

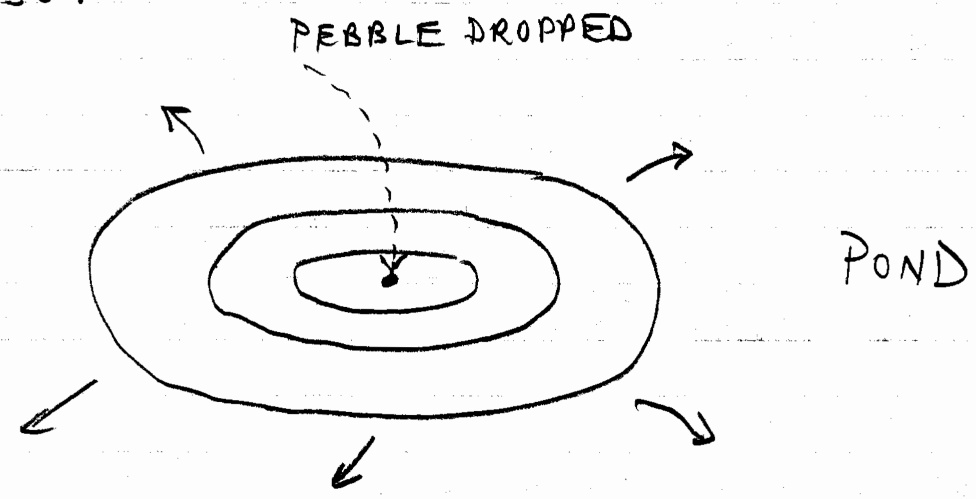
$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$\approx \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{\Delta f}{\Delta x} = \frac{\Delta y}{\Delta x}$$

a rate: change in y with respect to change in x

4/9/01 M Lecture

Example. Suppose a pebble is dropped into a calm pond and a circular wave spreads out like so:



Outline for Related-Rate Problems

(SEE HANDOUT)

SUGGESTIONS: (1) Read problem carefully.
(2) Draw figure where appropriate and label
(3) Then follow the steps below.

STEP 1. Decide what rate of change is desired and express it in "$\frac{dy}{dx}$" notation:

Find $\frac{dQ}{dt}$ when ______.

STEP 2. Decide what rates of change are given and express these data in "$\frac{dy}{dx}$" notation:

Given $\frac{dr}{dt} =$ ______ and $\frac{ds}{dt} =$ ______

when ______.

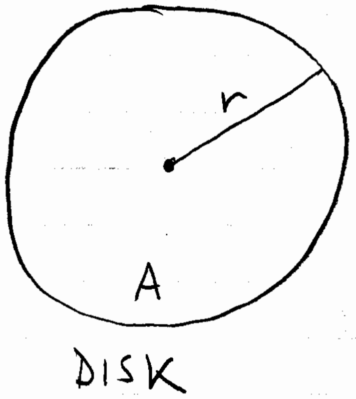
STEP 3. Find an equation relating Q, r, and s. You may have to use a figure or some geometric formula.

STEP 4. Differentiate, with respect to time t, the relation equation in STEP 3 (often by IMPLICIT DIFFERENTIATION) to obtain a relation among $\frac{dQ}{dt}$, $\frac{dr}{dt}$, $\frac{ds}{dt}$.

STEP 5. Put in values of r, s, Q and of $\frac{dr}{dt}, \frac{ds}{dt}$ corresponding to the instant when $\frac{dQ}{dt}$ is desired, and solve for $\frac{dQ}{dt}$.

Examples.

1. If the radius r of a circular disk is increasing at the rate of 3 inches per second, find the rate of increase of its area when $r=4$ inches.



$A = A(t)$ = area as a function of time

$r = r(t)$ = radius as a function of time

STEP 1. Find $\frac{dA}{dt}$ when $r=4$ in.

STEP 2. Given $\frac{dr}{dt} = 3$ in/sec when $r = 4$ in.

STEP 3. Formula relating A and r of a circle:

$$A = \pi r^2$$

STEP 4. IMPLICIT DIFFERENTIATION of $A=\pi r^2$ with respect to t where $A=A(t)$ and $r=r(t)$:

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2) \implies$$

$$\frac{d}{dt}(A) = \pi \frac{d}{dt}(r^2) \implies$$

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) \implies$$

$$\boxed{\frac{dA}{dt} = 2\pi r \frac{dr}{dt}}$$

Just like implicitly differentiating $x^2+y^2=1$ (where t is like x and A and r are like y):

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) \implies$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1) \implies$$

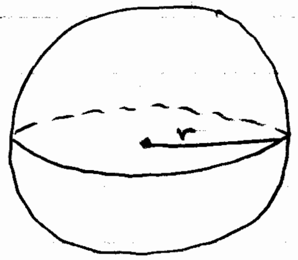
$$2x + 2y\frac{dy}{dx} = 0$$

<u>STEP 5</u>. When $r=4$, we have

$$\frac{dA}{dt} = 2\pi \underset{4}{r}\, \underset{3}{\frac{dr}{dt}} = 2\pi(4)(3) = \boxed{24\,\frac{\text{in}^2}{\text{sec}}}.$$

2. (HW Exercise 3, p. 272.)

If a snowball melts so that its surface area decreases at a rate of 1 cm²/min, find the rate at which the diameter decreases when the diameter is 10 cm.



SNOWBALL

$r = r(t) =$ radius

$D = D(t) =$ diameter

$A = A(t) =$ surface area

STEP 1. Find $\frac{dD}{dt}$ when $D = 10$ cm.

STEP 2. Given $\frac{dA}{dt} = -1$ cm²/min when $D = 10$ cm.

IMPORTANT:

E.g., $\frac{dA}{dt} = -1$: Surface area decreases at a rate of 1 cm²/min

E.g., $\frac{dA}{dt} = 1$: Surface area increases at a rate of 1 cm²/min

<u>STEP 3</u>. Formula relating A and D of a sphere:

FROM FRONT COVER OF TEXT UNDER "SPHERE" $\Rightarrow$

$$A = 4\pi r^2 \Rightarrow$$

$$D = 2r \text{ so } r = \frac{D}{2}$$

$$A = 4\pi\left(\frac{D}{2}\right)^2 \Rightarrow$$

$$A = \cancel{4}\pi \frac{D^2}{\cancel{4}} \Rightarrow$$

$$A = \pi D^2$$

<u>STEP 4</u>. IMPLICIT DIFFERENTIATION of $A = \pi D^2$ with respect to t where $A = A(t)$ and $D = D(t)$:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi D^2\right) \Rightarrow$$

$$\frac{d}{dt}(A) = \pi \frac{d}{dt}(D^2) \Rightarrow$$

$$\frac{dA}{dt} = \pi\left(2D\frac{dD}{dt}\right) \Rightarrow$$

$$\boxed{\frac{dA}{dt} = 2\pi D\frac{dD}{dt}}$$

STEP 5. When $D = 10$, we have

$$\frac{dA}{dt} = 2\pi D \frac{dD}{dt} \Rightarrow$$

WANT TO SOLVE FOR $\frac{dD}{dt}$

WHERE KNOW $\frac{dA}{dt} = -1$ and $D = 10$

$$\frac{dD}{dt} = \frac{1}{\underset{\substack{\uparrow \\ 10}}{2\pi D}} \cdot \underset{\substack{\uparrow \\ -1}}{\frac{dA}{dt}} = \frac{1}{2\pi(10)} \cdot (-1) = \boxed{-\frac{1}{20\pi} \ \frac{cm}{min}}$$

4/9/01
M
Lecture

$\therefore$ Diameter is decreasing at a rate of $\frac{1}{20\pi} \frac{cm}{min}$.