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Section 3.6. Implicit Differentiation.

Examples of Explicit and Implicit Functions

y is an explicit function of x (forming an "explicit equation")	y is an implicit function of x (forming an "implicit equation")
$y = 3(x-2)^2 + 5$	$x = 3(y-2)^2 + 5$
$y = e^{x^2+3}$	$x = y + e^y$
$y = \sin x^3$	$y^3 - 2x^2y^2 + 1 = 0$
$y = \frac{1}{x}$	$\frac{y}{y^2+x} = 7$

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Explicit:

- y is isolated and placed on one side of "="
- expression in x only is on the other side of "="
- y is "solved for"

Implicit:

- usually x's and y's are "mixed together" <u>or</u> y is raised to some power > 1 and there is more than one y term
- y cannot be "solved for" <u>or</u> solving for y will

lead to more than one function, as with

$$x^2+y^2=1 \Rightarrow$$

$$y = \pm\sqrt{1-x^2} \Rightarrow$$

$$y=\sqrt{1-x^2}, \quad y=-\sqrt{1-x^2}$$

2 different functions

NOTES:

1. With $x^2+y^2=1$: y is an implicit function of x since one <u>cannot</u> solve for y <u>explicitly</u>

$$x^2+y^2=1 \Rightarrow y^2 = 1-x^2$$

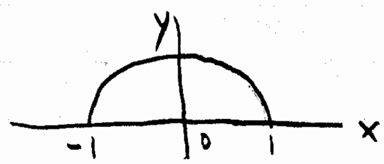
$$\Rightarrow y = \pm\sqrt{1-x^2}$$

So y is not given explicitly, or definitely, or unambiguously.

Either y is the function

$$y = \sqrt{1-x^2}$$

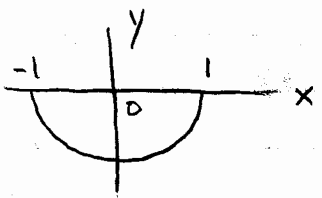
with graph



or y is the function

$$y = -\sqrt{1-x^2}$$

with graph



2. But with $x^3+y^3=1$; y is an explicit function of x since one <u>can</u> solve for y <u>explicitly</u>

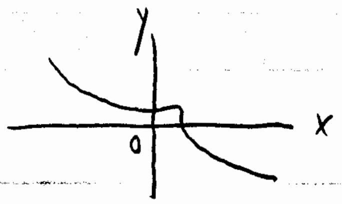
$$x^3+y^3=1 \Rightarrow y^3 = 1-x^3$$

$$\Rightarrow y = \sqrt[3]{1-x^3}$$

So, y is given explicitly as the function

$$y = \sqrt[3]{1-x^3}$$

with graph



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Implicit Differentiation

<u>Defn</u>. <u>Implicit differentiation</u> is the method used to find the derivative of an <u>implicit function of x</u>, usually by taking the derivative, $\frac{d}{dx}$, of both sides of the <u>implicit equation in x and y</u>.

<u>NOTES</u>: (1) We view y as a function (unknown) of x:

$$y = y(x)$$

(2) We then use the chain rule on y:

$$\frac{d}{dx} f(y(x)) = f'(y(x)) \cdot y'(x)$$

Examples.

1. Let $x^2+y^2=4$. Find y'.

y is an implicit function of x
and $x^2+y^2=4$ is an implicit equation.
We can approach this problem IN TWO DIFFERENT WAYS.

① IMPLICIT DIFFERENTIATION:

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(4) \Rightarrow$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(4) \Rightarrow$$

$$2x + \underbrace{2yy'}_{} = 0 \Rightarrow$$

$$\frac{d}{dx}(y(x))^2 = 2(y(x))^1 \cdot y'(x) = 2yy'$$

CHAIN RULE: $\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$

(Specifically: $\frac{d}{dx}(g(x))^n = n(g(x))^{n-1}\cdot g'(x)$)

Inside function: $g(x)=y(x)$

$g'(x)=y'(x)=y'$

Outside function: $f(x)=x^2$

$f'(x)=2x^1$

$f'(g(x))=f'(y(x))=2y(x)=2y$

$$2x+2yy'=0 \Rightarrow$$

SOLVE FOR y'

$$2yy' = -2x \Rightarrow$$

$$y' = \frac{-\cancel{2}x}{\cancel{2}y} \Rightarrow$$

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$$\boxed{y' = -\frac{x}{y}} \quad \left(\text{or } \frac{dy}{dx} = -\frac{x}{y}\right)$$

② BREAK UP $x^2+y^2=4$ INTO TWO FUNCTIONS AND THEN FIND DERIVATIVE OF EACH:

$$x^2+y^2=4 \Rightarrow y^2 = 4-x^2$$

$$\Rightarrow y = \pm\sqrt{4-x^2}$$

ⓐ $y = \sqrt{4-x^2}$:

$$y' = \frac{d}{dx}\left(\sqrt{4-x^2}\right)$$

$$= \frac{d}{dx}(4-x^2)^{1/2}$$

CHAIN RULE: $\frac{d}{dx}(g(x))^{1/2} = \frac{1}{2}(g(x))^{-\frac{1}{2}} \cdot g'(x)$

$$= \frac{g'(x)}{2\sqrt{g(x)}}$$

$$= \frac{1}{2}(4-x^2)^{-1/2} \cdot \frac{d}{dx}(4-x^2)$$

$$= \frac{1}{\cancel{2}}(4-x^2)^{-1/2}(-\cancel{2}x)$$

$$= -\frac{x}{\sqrt{4-x^2}}$$

$y = \sqrt{4-x^2}$

$$= -\frac{x}{y}$$ ✓

(b) $y = -\sqrt{4-x^2}$

$$y' = \frac{d}{dx}\left(-\sqrt{4-x^2}\right)$$

$$= -\frac{d}{dx}(4-x^2)^{1/2}$$

$$= \cancel{-}\frac{1}{\cancel{2}}(4-x^2)^{-1/2}(\cancel{-}\cancel{2}x)$$

$$= \frac{x}{\sqrt{4-x^2}}$$

$y = -\sqrt{4-x^2} \Rightarrow -y = \sqrt{4-x^2}$

$$= -\frac{x}{y}$$ ✓

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2. (HW Exercise 1, p. 245.)

Let $x^2+3x+xy=5$. Find y' in 2 ways.

(a) IMPLICIT DIFFERENTIATION:

$$\frac{d}{dx}\left(x^2+3x+xy\right)=\frac{d}{dx}(5) \implies$$

$$\frac{d}{dx}(x^2)+3\frac{d}{dx}(x)+\frac{d}{dx}(xy)=\frac{d}{dx}(5) \implies$$

$$2x+3(1)+\underbrace{\left[\left(\frac{d}{dx}x\right)y+x\left(\frac{d}{dx}y\right)\right]}=0 \implies$$

$$\frac{d}{dx}\left(x\,y(x)\right)=\left[\left(\frac{d}{dx}x\right)y(x)+x\left(\frac{d}{dx}y(x)\right)\right]$$

PRODUCT RULE:

$$\frac{d}{dx}\left(f(x)g(x)\right)=\left(\frac{d}{dx}f(x)\right)g(x)+f(x)\left(\frac{d}{dx}g(x)\right)$$

$$f(x)=x, \quad g(x)=y(x)$$

$$2x+3+\left[(1)y+x\frac{dy}{dx}\right]=0 \implies$$

$$2x+3+y+xy'=0 \implies$$

SOLVE FOR y'

$$xy' = -2x - 3 - y \Rightarrow$$

$$\boxed{y' = \frac{-2x-3-y}{x} = -2 - \frac{3}{x} - \frac{y}{x}}$$

(b) SOLVE $x^2+3x+xy=5$ EXPLICITLY FOR y AND DIFFERENTIATE USUAL WAY:

$$x^2+3x+xy=5 \Rightarrow$$

$$xy = 5 - x^2 - 3x \Rightarrow$$

$$y = \frac{5-x^2-3x}{x} \Rightarrow$$

$$y = \frac{5}{x} - \frac{x^2}{x} - \frac{3\cancel{x}}{\cancel{x}} \Rightarrow$$

$$y = 5 \cdot \frac{1}{x} - \frac{x \cdot \cancel{x}}{\cancel{x}} - 3 \Rightarrow$$

$$y = 5x^{-1} - x - 3$$

$$\therefore\ y' = \frac{d}{dx}\left(5x^{-1} - x - 3\right)$$

$$= 5(-x^{-2}) - 1 - 0$$

$$= -5x^{-2} - 1$$

$$= -5 \cdot \frac{1}{x^2} - 1$$

$$= \boxed{-\frac{5}{x^2} - 1}$$

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(c) To see that the solutions in (a) and (b) are the same, substitute

$$y = \frac{5}{x} - x - 3 \quad \text{(from (b))}$$

into

$$y' = \frac{-2x - 3 - y}{x} \quad \text{(from (a))}.$$

So,

$$y' = \frac{-2x - 3 - \left(\frac{5}{x} - x - 3\right)}{x}$$

$$= \frac{-2x - \cancel{3} - \frac{5}{x} + x + \cancel{3}}{x}$$

$$= \frac{-x - \frac{5}{x}}{x}$$

$$= \frac{-x}{x} - \frac{\frac{5}{x}}{x}$$

$$= -1 - \frac{5}{x} \cdot \frac{1}{x} \quad \left(\frac{\frac{5}{x}}{\frac{x}{1}} = \frac{5}{x} \div \frac{x}{1} = \frac{5}{x} \cdot \frac{1}{x} \right)$$

$$= -1 - \frac{5}{x^2}$$

$$= -\frac{5}{x^2} - 1 \quad \checkmark$$

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3. (HW Exercise 8, p. 245.)

Let $x\sin y + \cos 2y = \cos y$. Find $\frac{dy}{dx}$ $[=y']$

$$\frac{d}{dx}\left(x\sin y + \cos 2y\right) = \frac{d}{dx}(\cos y) \Rightarrow$$

$$\frac{d}{dx}(x\sin y) + \frac{d}{dx}(\cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow$$

$$\underbrace{\left[\left(\frac{d}{dx}x\right)\sin y + x\left(\frac{d}{dx}\sin y\right)\right]} + \frac{d}{dx}(\cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow$$

$$\frac{d}{dx}(x\sin y(x)) = \left[\left(\frac{d}{dx}x\right)\sin y(x) + x\left(\frac{d}{dx}\sin y(x)\right)\right]$$

PRODUCT RULE:

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

$$f(x) = x, \quad g(x) = \sin y(x)$$

$$[(1)\sin y + x\underbrace{(\cos y)y'}] + \underbrace{2(-\sin 2y)y'} = \underbrace{(-\sin y)y'} \Rightarrow$$

$$\frac{d}{dx}(\sin y(x)) = (\cos y(x))\cdot y'(x)$$

CHAIN RULE

$$\frac{d}{dx}(\cos 2y(x)) = 2(-\sin y(x))\cdot y'(x)$$

APPLICATION OF FORMULA

$$\frac{d}{dx}\cos 2x = 2(-\sin 2x)$$

AND CHAIN RULE

$$\frac{d}{dx}(\cos y(x)) = (-\sin y(x))\cdot y'(x)$$

CHAIN RULE

$\sin y + xy'\cos y - 2y'\sin 2y = -y'\sin y \Rightarrow$

SOLVE FOR y'

$xy'\cos y - 2y'\sin 2y + y'\sin y = -\sin y \Rightarrow$

$y'(x\cos y - 2\sin 2y + \sin y) = -\sin y \Rightarrow$

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$$\boxed{y' = \frac{-\sin y}{x\cos y - 2\sin 2y + \sin y}}$$

4. (HW Exercise 12, p. 245.)

Find an equation to the tangent line to the curve

$$\frac{x^2}{9} + \frac{y^2}{36} = 1 \quad [\text{ellipse}]$$

at the point

$$(x, y) = (-1, 4\sqrt{2})$$

<u>Notation</u>:

Slope of tangent line to curve $\frac{x^2}{9} + \frac{y^2}{36} = 1$

at point $(-1, 4\sqrt{2}) = m_{tan}$

$$= \left.\frac{dy}{dx}\right|_{(-1, 4\sqrt{2})}$$

<u>Note</u>: $\frac{x^2}{9} + \frac{y^2}{36} = 1$ is an implicit equation $\Rightarrow$ must use implicit differentiation to find $\frac{dy}{dx}$ and then need to substitute values for x and y in the formula for $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(\frac{x^2}{9}+\frac{y^2}{36}\right)=\frac{d}{dx}(1) \Rightarrow$$

$$\frac{d}{dx}\left(\frac{1}{9}\cdot x^2+\frac{1}{36}\cdot y^2\right)=\frac{d}{dx}(1) \Rightarrow$$

$$\frac{1}{9}\frac{d}{dx}(x^2)+\frac{1}{36}\frac{d}{dx}(y^2)=\frac{d}{dx}(1) \Rightarrow$$

$$\frac{1}{9}(2x)+\frac{1}{36}(2yy')=0 \Rightarrow$$

$$\frac{2}{9}x+\frac{1}{18}yy'=0 \Rightarrow$$

$$\frac{1}{18}yy'=-\frac{2}{9}x \Rightarrow$$

$$y'=\frac{\frac{2}{9}x}{\frac{1}{18}y} \Rightarrow$$

$$y'=\frac{\frac{2}{9}}{\frac{1}{18}}\cdot\frac{x}{y} \Rightarrow$$

$$y'=\frac{2}{\cancel{9}}\cdot\frac{\overset{2}{\cancel{18}}}{1}\cdot\frac{x}{y} \Rightarrow$$

$$y'=\frac{4x}{y} \Rightarrow$$

$$\frac{dy}{dx} = \frac{4x}{y}$$

$$\therefore \left.\frac{dy}{dx}\right|_{(x,y)=(-1,\,4\sqrt{2})} = \frac{\cancel{4}(-1)}{\cancel{4}\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\therefore m_{tan} = -\frac{1}{\sqrt{2}}, \quad (x_1, y_1) = (-1, 4\sqrt{2})$$

$$y - y_1 = m_{tan}(x - x_1) \implies$$

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$$\boxed{y - 4\sqrt{2} = -\frac{1}{\sqrt{2}}(x+1)}$$

EQ. OF TANGENT LINE