Section 3.6. Implicit Differentiation.

Examples of Explicit and Implicit Functions

<table>
<thead>
<tr>
<th>Explicit Function of x</th>
<th>Implicit Function of x</th>
</tr>
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<tbody>
<tr>
<td>y = 3(x - 2)^3 + 5</td>
<td>x = 3(y - 2)^3 + 5</td>
</tr>
<tr>
<td>y = e^{x^2 + 3}</td>
<td>x = y + e^y</td>
</tr>
<tr>
<td>y = \sin(x^2)</td>
<td>y^3 - 2x^2y^2 + 1 = 0</td>
</tr>
<tr>
<td>y = \frac{1}{x}</td>
<td>\frac{y}{y^3 + x} = 7</td>
</tr>
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</table>

- y is isolated and placed on one side of "="
- Expression in x only is on the other side of "="
- y is "solved for"
- Usually x's and y's are "mixed together" or y is raised to some power > 1 and there is more than one y term
- y cannot be "solved for" or solving for y will
lead to more than one function, as with

\[ x^2 + y^2 = 1 \Rightarrow \]

\[ y = \pm \sqrt{1-x^2} \Rightarrow \]

\[ y = \sqrt{1-x^2}, \ y = -\sqrt{1-x^2} \]

2 different functions

NOTES:

1. With \( x^2 + y^2 = 1 \): \( y \) is an implicit function of \( x \) since one cannot solve for \( y \) explicitly.

\[ x^2 + y^2 = 1 \Rightarrow y^2 = 1-x^2 \]

\[ \Rightarrow y = \pm \sqrt{1-x^2} \]

So \( y \) is not given explicitly, or definitely, or unambiguously.
Either \( y \) is the function \( y = \sqrt{1 - x^2} \) with graph

\[ \text{graph} \]

or \( y \) is the function \( y = -\sqrt{1 - x^2} \) with graph

\[ \text{graph} \]

2. But with \( x^3 + y^3 = 1 \); \( y \) is an explicit function of \( x \) since one can solve for \( y \) explicitly

\[ x^3 + y^3 = 1 \Rightarrow y^3 = 1 - x^3 \Rightarrow y = \sqrt[3]{1 - x^3} \]
So, $y$ is given explicitly as the function

$$y = \frac{1}{2}\sqrt{1-x^3}$$

with graph

[Graph of the function $y = \frac{1}{2}\sqrt{1-x^3}$]
**Implicit Differentiation**

**Definition.** Implicit differentiation is the method used to find the derivative of an implicit function of \( x \), usually by taking the derivative, \( \frac{dy}{dx} \), of both sides of the implicit equation in \( x \) and \( y \).

**NOTES:**
1. We view \( y \) as a function (unknown) of \( x \):
   
   \[ y = y(x) \]

2. We then use the chain rule on \( y \):
   
   \[ \frac{dy}{dx} f(y(x)) = f'(y(x)) \cdot y'(x) \]
Examples.

1. Let \( x^2 + y^2 = 4 \). Find \( y' \).

\( y \) is an implicit function of \( x \) and \( x^2 + y^2 = 4 \) is an implicit equation. We can approach this problem in two different ways:

1. **Implicit Differentiation:**

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4) \Rightarrow \\
\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4) \Rightarrow \\
2x + 2yy' = 0 \Rightarrow \\
\frac{d}{dx}(y(x))^2 = 2(y(x))^1 \cdot y'(x) = 2yy' \\
\text{Chain Rule: } \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \\
\text{Specifically: } \frac{d}{dx} (y(x)) = y'(x) = 2yy' \\
\text{Inside function: } g(x) = y(x) \\
\text{Outside function: } f(x) = x^2 \\
f'(x) = 2x \\
\frac{d}{dx} (y(x)) = f'(y(x)) = 2yy' \\
2x + 2yy' = 0 \Rightarrow \\
Solve for \ y' \]
\[2yy' = -2x \implies y' = \frac{-x}{2y}\]

\[y' = -\frac{x}{y}\]

\[
\left(\text{or}\quad \frac{dy}{dx} = -\frac{x}{y}\right)
\]

**2** Break up \(x^2 + y^2 = 4\) into two functions and then find derivative of each:

\[x^2 + y^2 = 4 \implies y^2 = 4 - x^2 \implies y = \pm \sqrt{4 - x^2}\]

**6**

\[y = \sqrt{4 - x^2}\]

\[y' = \frac{d}{dx} \left(\sqrt{4 - x^2}\right)\]

\[= \frac{1}{2} (4 - x^2)^{-\frac{1}{2}} \cdot (-2x)\]

\[= \frac{-x}{\sqrt{4 - x^2}}\]
\[ y = \sqrt[4]{4-x^2} \]

\[ y' = \frac{d}{dx} \left( -\sqrt[4]{4-x^2} \right) \]

\[ = - \frac{1}{4} \left( 4-x^2 \right)^{-\frac{3}{4}} \cdot (-2x) \]

\[ = \frac{x}{2} \left( 4-x^2 \right)^{-\frac{3}{4}} \]

\[ \Rightarrow -y = \sqrt[4]{4-x^2} \]

\[ = -\frac{x}{\sqrt[4]{4-x^2}} \]
Let $x^2 + 3x + xy = 5$. Find $y'$ in 2 ways.

(a) Implicit Differentiation:

\[
\frac{d}{dx}(x^2 + 3x + xy) = \frac{d}{dx}(5) \Rightarrow
\]

\[
\frac{d}{dx}(x^2) + 3 \frac{d}{dx}(x) + \frac{d}{dx}(xy) = \frac{d}{dx}(5) \Rightarrow
\]

\[
2x + 3(1) + \left[\left(\frac{d}{dx}(x)\right)y + x\left(\frac{d}{dx}y\right)\right] = 0 \Rightarrow
\]

\[
\frac{d}{dx}(xy(x)) = \left[\left(\frac{d}{dx}(x)\right)y(x) + x\left(\frac{d}{dx}y(x)\right)\right]
\]

Product Rule:

\[
\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)
\]

$f(x) = x$, $g(x) = y(x)$

\[
2x + 3 + \left[(1)y + x \frac{d}{dx}y\right] = 0 \Rightarrow
\]

\[
2x + 3 + y + xy' = 0 \quad \text{Solve for } y'
\]
\( xy' = -2x - 3 - y \quad \Rightarrow \)

\[
y' = \frac{-2x - 3 - y}{x} = -2 - \frac{3}{x} - \frac{y}{x}
\]

(1) Solve \( x^2 + 3x + xy = 5 \) explicitly for \( y \) and differentiate usual way:

\[ x^2 + 3x + xy = 5 \quad \Rightarrow \]

\[ xy = 5 - x^2 - 3x \quad \Rightarrow \]

\[ y = \frac{5 - x^2 - 3x}{x} \quad \Rightarrow \]

\[ y = \frac{5}{x} - \frac{x^2}{x} - \frac{3x}{x} \quad \Rightarrow \]

\[ y = 5x^{-1} - x - 3 \quad \Rightarrow \]

\[
y = 5x^{-1} - x - 3
\]

\[
y' = \frac{d}{dx} (5x^{-1} - x - 3) = 5(-x^{-2}) - 1 - 0
\]
\[ -2 \frac{9}{2} \]
\[ = -5x^{-2} - 1 \]
\[ = -5 \frac{1}{x^2} - 1 \]
\[ = -\frac{5}{x^2} - 1 \]

(c) To see that the solutions in (a) and (b) are the same, substitute
\[ y = \frac{5}{x} - x - 3 \] (from (b))
into
\[ y' = \frac{-2x - 3 - y}{x} \] (from (a)).

So,
\[ y' = \frac{-2x - 3 - \left( \frac{5}{x} - x - 3 \right)}{x} \]
\[ = \frac{-2x - 3 - \frac{5}{x} + x + 3}{x} \]
\[
\begin{align*}
&= -\frac{x - \frac{5}{x}}{x} \\
&= -\frac{x}{x} - \frac{5}{x} \\
&= -1 - \frac{5}{x^2} \cdot \frac{1}{x} \\
&= -1 - \frac{5}{x^3} \\
&= -\frac{5}{x^2} - 1
\end{align*}
\]
Let \( x \sin y + \cos 2y = \cos y \). Find \( \frac{dy}{dx} \) [\( y' \)].

\[
\frac{d}{dx}(x \sin y + \cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow
\]
\[
\frac{d}{dx}(x \sin y) + \frac{d}{dx}(\cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow
\]
\[
\left[ (\frac{d}{dx})(x \sin y) + x \left( \frac{d}{dx}(\sin y) \right) \right] + \frac{d}{dx}(\cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow
\]
\[
\frac{d}{dx}(x \sin y(x)) = \left[ (\frac{d}{dx})(\sin y(x)) + x \left( \frac{d}{dx}(\sin y(x)) \right) \right]
\]
PRODUCT RULE:
\[
\frac{d}{dx}(f(x)g(x)) = \left( \frac{df}{dx} \right)g(x) + f(x)\left( \frac{dg}{dx} \right)
\]
\[
f(x) = x, \quad g(x) = \sin y(x)
\]
\[
\left[ (\sin y + x \left( \cos y \right) y') \right] + 2(\sin 2y) y' = (-\sin y) y' \Rightarrow
\]
\[
\frac{d}{dx}(\sin y(x)) = \left( \cos y(x) \right) y'(x)
\]
CHAIN RULE
\[
\frac{d}{dx}(\cos 2y) = 2(-\sin y(x)) y'(x)
\]
APPLICATION OF FORMULA
\[
\frac{d}{dx}(\sin 2y) = 2(-\sin y(x)) y'(x)
\]
AND CHAIN RULE
\[
\frac{d}{dx}(\sin 2y) = \left( \cos 2y(x) \right) y'(x)
\]
CHAIN RULE
\[
\begin{align*}
\sin y + xy' \cos y - 2y' \sin 2y &= -y' \sin y \\
\Rightarrow \\
xy' \cos y - 2y' \sin 2y + y' \sin y &= -y' \\
\Rightarrow \\
y' \left( x \cos y - 2 \sin 2y + \sin y \right) &= -y' \\
\Rightarrow \\
y' &= \frac{-y'}{x \cos y - 2 \sin 2y + \sin y}
\end{align*}
\]
4. (HW Exercise 12, p. 245.)

Find an equation to the tangent line to the curve

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

[elliptic]

at the point

$$(x, y) = (-1, 4\sqrt{2})$$

Notation:

Slope of tangent line to curve $\frac{x^2}{9} + \frac{y^2}{36} = 1$

at point $(-1, 4\sqrt{2}) = mn tan$

$$= \frac{dy}{dx} \bigg|_{(-1, 4\sqrt{2})}$$

Note: $\frac{x^2}{9} + \frac{y^2}{36} = 1$ is an implicit equation $\Rightarrow$ must use implicit differentiation to find $\frac{dy}{dx}$ and then need to substitute values for $x$ and $y$ in the formula for $\frac{dy}{dx}$. 
\[
\frac{d}{dx} \left( \frac{x^2}{9} + \frac{y^2}{36} \right) = \frac{d}{dx} (1) \quad \Rightarrow \\
\frac{d}{dx} \left( \frac{1}{4} \cdot x^2 + \frac{1}{36} \cdot y^2 \right) = \frac{d}{dx} (1) \quad \Rightarrow \\
\frac{1}{4} \frac{d}{dx} (x^2) + \frac{1}{36} \frac{d}{dx} (y^2) = \frac{d}{dx} (1) \quad \Rightarrow \\
\frac{1}{4} (2x) + \frac{1}{36} (2y y') = 0 \quad \Rightarrow \\
\frac{2}{9} x + \frac{1}{18} y y' = 0 \quad \Rightarrow \\
\frac{1}{18} y y' = - \frac{2}{9} x \quad \Rightarrow \\
y' = \frac{\frac{2}{9} x}{\frac{1}{18} y} \quad \Rightarrow \\
y' = \frac{\frac{3}{9} \cdot x}{\frac{1}{18} y} \quad \Rightarrow \\
y' = \frac{\frac{2}{3} \cdot \frac{18}{1} \cdot x}{y} \quad \Rightarrow \\
y' = \frac{4x}{y} \quad \Rightarrow
\[
\frac{dy}{dx} = \frac{4x}{y}
\]

\[
\frac{dy}{dx} \bigg|_{(x,y) = (-1, 4\sqrt{2})} = \frac{4(-1)}{4\sqrt{2}} = -\frac{1}{\sqrt{2}}
\]

\[
\tan \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \arctan\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}
\]

\[
(x_1, y_1) = (-1, 4\sqrt{2})
\]

\[
y - y_1 = \tan \theta (x - x_1) \Rightarrow
\]

\[
y - 4\sqrt{2} = -\frac{1}{\sqrt{2}}(x + 1)
\]

**EQ. OF TANGENT LINE**