Section 3.4. Derivatives of Trigonometric Functions.

REVIEW OF TRIG FUNCTIONS

There are six basic trig functions and their inverses:

\[
\begin{align*}
\sin t & \quad (\text{sine}) & \csc t & = \frac{1}{\sin t} & \quad (\text{cosecant}) \\
\cos t & \quad (\text{cosine}) & \sec t & = \frac{1}{\cos t} & \quad (\text{secant}) \\
\tan t & \quad (\text{tangent}) & \cot t & = \frac{1}{\tan t} & \quad (\text{cotangent})
\end{align*}
\]

\[
\begin{align*}
\sin^{-1} t & = \arcsin t & \csc^{-1} t \\
\cos^{-1} t & = \arccos t & \sec^{-1} t \\
\tan^{-1} t & = \arctan t & \cot^{-1} t
\end{align*}
\]

(\text{\(t\) IN RADIANS, not in degrees!!})
1. \( \sin^{-1}(\sin t) = t \) \( \sin(\sin^{-1} t) = t \)

2. \( \cos^{-1}(\cos t) = t \) \( \cos(\cos^{-1} t) = t \)

3. \( \tan^{-1}(\tan t) = t \) \( \tan(\tan^{-1} t) = t \)
4. $t$ must be in radians when using the derivative formulas for the first six trig functions.

**REASON:** Derivative formula for $\sin t$ is obtained from limit definition of derivative

$$f'(t) = \lim_{h \to 0} \frac{\sin (t+h) - \sin(t)}{h}$$

which, in turn, uses the fact that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1,$$

which, in turn, assumes $t$ is in radians.

Derivative formula for $\cos t$ comes from fact that

$$\lim_{t \to 0} \frac{\sin t}{t} \Rightarrow \lim_{t \to 0} \frac{\cos t - 1}{t} = 0,$$

Derivative formulas for $\tan t$, $\sec t$, $\csc t$, and $\cot t$ come from derivative formulas for $\sin t$ and $\cos t$ and from using quotient rule.
5. Will use $t$, $x$, $\theta$ (Greek letter "theta"), etc., as the independent variable whose units are radians.
5. IDENTITIES YOU SHOULD KNOW

\[ \sin^2 x + \cos^2 x = 1 \quad (\text{Fundamental Identity}) \]

\[ \tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \]

\[ \csc x = \frac{1}{\sin x} \quad \tan^2 x + 1 = \sec^2 x \]

\[ \sec x = \frac{1}{\cos x} \]

3. \[ \sin (x + k(2\pi)) = \sin x, \quad \cos (x + k(2\pi)) = \cos x \quad (k = 0, \pm 1, \pm 2, \ldots) \]

\[ e.g., \quad \sin (x + 2\pi) = \sin x \]

\[ \cos (x - 2\pi) = \cos x \]

\[ x + (-1)(2\pi) \]

\[ \sin (x + (\pi)) = \sin x \]

\[ x + \epsilon (\pi) \]

\[ e.g., \quad \sin x = 1 \quad \text{for what values of } x? \]
\[ x = \frac{\pi}{2} + k(2\pi), \quad k = 0, \pm 1, \pm 2, \ldots \]

\[ \cos x = -\frac{1}{2} \quad \text{for what values of } x? \]
\[ x = \frac{\sqrt{3}}{2} t + k(2\pi), \ k = 0, \pm 1, \pm 2, \ldots \]

*(SEE HW Exercise 25, p. 236.)*
Derivatives of Trig Functions

**MEMORIZE**

\[
\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x
\]

(x in radians!)

Do not worry about how these derivatives were derived.

However, know how to derive \( \frac{d}{dx}(\tan x) \) using the quotient rule:

\[
\left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2}
\]

\[
\tan x = \frac{\sin x}{\cos x} \implies \frac{d}{dx}(\tan x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right)
\]

\[
= \frac{\left( \frac{d}{dx}\sin x \right) \cos x - \sin x \left( \frac{d}{dx}\cos x \right)}{(\cos x)^2}
\]

\[
= \frac{(\cos x) \cos x - \sin x (-\sin x)}{(\cos x)^2}
\]
\[
\begin{align*}
\frac{\cos^2 x + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
&= \left(\frac{1}{\cos x}\right)^2 \\
&= (\sec x)^2 \\
&= \sec^2 x
\end{align*}
\]
Also be able to derive

\[
\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)
\]
\[
\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)
\]
\[
\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)
\]
Examples.

1. (HW Exercise 1, p. 225.)

\[ y = \sin x + \cos x \]

\[ y' = \frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x) = \cos x - \sin x \]

2. (HW Exercise 4, p. 225.)

\[ y = e^x \sin x \]

\[ y' = \frac{d}{dx}(e^x \sin x) \]

\[ \text{Use Product Rule: } (fg)' = fg' + f'g \]

\[ = \left( \frac{d}{dx} e^x \right) \sin x + e^x \left( \frac{d}{dx} \sin x \right) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x) \]
\[ y = xcscx \]

\[ y' = \frac{d}{dx} (xcscx) = \frac{d}{dx} (x \cdot \frac{1}{\sin x}) \]

\[ = \frac{d}{dx} \left( \frac{x}{\sin x} \right) \]

- USE THE QUOTIENT RULE: \( \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \)

\[ = \left( \frac{\frac{d}{dx} x}{\sin x} - x \left( \frac{\frac{d}{dx} \sin x}{(\sin x)^2} \right) \right) \]

\[ = \frac{(1) \sin x - x \cos x}{\sin^2 x} \]

\[ = \frac{\sin x - x \cos x}{\sin^2 x} \]

\[ = \frac{\sin x}{\sin^2 x} - \frac{x \cos x}{\sin^2 x} \]

\[ = \frac{1}{\sin x} - x \cdot \frac{\cos x}{\sin x} \cdot \cot x \]

\[ = \csc x \cdot \left( 1 - x \cot x \right) \]
4. (HW Exercise 12, p. 226.)

\[ y = x \sin x \cos x \]

\[ y' = \frac{d}{dx} \left[ (x \sin x) \cos x \right] \]

**Use Product Rule Twice:**

\[(fg)' = f'g + fg'\]

\[= \left( \frac{d}{dx} (x \sin x) \right) \cos x + x \sin x \left( \frac{d}{dx} \cos x \right) \]

\[= \left[ \frac{d}{dx} (x \sin x) \right] \cos x + x \sin x (-\sin x) \]

\[= \left[ \left( \frac{d}{dx} x \right) \sin x + x \left( \frac{d}{dx} \sin x \right) \right] \cos x - x \sin^2 x \]

\[= \left[ (1) \sin x + x \cos x \right] \cos x - x \sin^2 x \]

\[= \sin x \cos x + x \cos^2 x - x \sin^2 x \]