Lecture

$\boxed{\text{CHAPTER 3}}$ Differentiation.

3/19/01 M lecture

We will now learn about different RULES OF DIFFERENTIATION (= rules of finding derivatives of functions). These actually come from

(1) the <u>limit definition of derivative</u>

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(2) the <u>limit laws</u> (Sect. 2.3), e.g.,

$$\begin{aligned}\lim_{x \to 1} (5x^2 + 2x^3) &= \lim_{x \to 1} 5x^2 + \lim_{x \to 1} 2x^3 \\ &= 5 \lim_{x \to 1} x^2 + 2 \lim_{x \to 1} x^3 \\ &= 5(1)^2 + 2(1)^3 \\ &= 7\end{aligned}$$

We will set aside the limit definition of derivative and start memorizing formulas and rules that come from it.

Lecture

Section 3.1. Derivatives of Polynomials and Exponential Functions.

Notation: If we are given, say, the function $f(x) = x^2$, we will write for its derivative

$$f'(x) = \frac{d}{dx}(x^2) \quad \left(\text{or } \frac{d}{dx} x^2 \right)$$

Power Rule

Let $f(x) = x$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \underset{\substack{f(x)=x \Rightarrow \\ f(\square)=\square \Rightarrow \\ f(x+h)=x+h}}{=} \lim_{h \to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 = \boxed{1 \cdot x^0}$$

Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$f(x) = x^2 \Rightarrow$
$f(\square) = (\square)^2 \Rightarrow$
$f(x+h) = (x+h)^2$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{h(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x+h) = 2x + 0 = 2x = \boxed{2x^1}$$

New notation:
$f'(x) \overset{\text{def}}{=} \frac{d}{dx}(f(x))$
$f''(x) \overset{\text{def}}{=} \frac{d}{dx}(f'(x)) = \frac{d^2}{dx^2}(f(x))$
etc.

It can be shown that, in general,

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}, \quad n \text{ any real number!}}$$

POWER RULE

<u>Examples.</u>

1. $\frac{d}{dx}(x^e) = \boxed{e x^{e-1}}$ $\quad (e = 2.71828\ldots)$

2. $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2} x^{\frac{1}{2}-1} = \boxed{\frac{1}{2} x^{-1/2}}$

PUT IN FRACTIONAL FORM

$$= \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \boxed{\frac{1}{2\sqrt{x}}}$$

<u>Recall</u>: $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.
E.g., $x^{3/2} = \sqrt{x^3} = (\sqrt{x})^3$.

3. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)\,x^{-1-1} = \boxed{-x^{-2}} = \boxed{-\frac{1}{x^2}}$

PUT IN NEGATIVE EXPONENT FORM

<u>Recall</u>: $x^{-n} = \frac{1}{x^n}$.

4. $\frac{d}{dx}(1) = \frac{d}{dx}(x^0) = 0 \cdot x^{0-1} = 0 \cdot x^{-1} = \boxed{0}$

<u>Recall</u>: $x^0 = 1$.

5. $\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = \boxed{1}$

6. $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = \boxed{2x}$

7. $\frac{d}{dx}(x^3) = 3x^{3-1} = \boxed{3x^2}$

Summary of some derivatives you may forget:

Put after # 1 and 2 below

$\frac{d}{dx}(1) = 0$	$\frac{d}{dx}(k) = 0$, k any constant E.g., $\frac{d}{dx}(2) = 0$
$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(kx) = k$, k any constant E.g., $\frac{d}{dx}(3x) = 3$

SUM / DIFFERENCE

Sum / Difference and Constant Multiple Rules

1. E.g., $\frac{d}{dx}(x^2 + x^3) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) = \boxed{2x + 3x^2}$

E.g., $\frac{d}{dx}\left(\frac{1}{x} - 1\right) = \frac{d}{dx}\left(\frac{1}{x}\right) - \frac{d}{dx}(1)$

$= \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(1)$

$= -x^{-2} - 0$

$= \boxed{-\frac{1}{x^2}}$

CONSTANT MULTIPLE

2. E.g., $\frac{d}{dx}\left(\frac{1}{2}x^4\right) = \frac{1}{2}\cdot\frac{d}{dx}(x^4) = \frac{1}{2}(4x^3) = \boxed{2x^3}$

E.g., $\frac{d}{dx}(2) = \frac{d}{dx}(2\cdot 1) = 2\cdot\frac{d}{dx}(1) = 2(0) = \boxed{0}$

E.g., $\frac{d}{dx}(3x) = 3\cdot\frac{d}{dx}(x) = 3(1) = \boxed{3}$

NOW WE PUT THESE RULES TOGETHER TO DIFFERENTIATE POLYNOMIAL FUNCTIONS

$\Downarrow$

Derivatives of Polynomials

Like 6 on p. 197

Example. (HW Exercise 10, p. 199.)

Differentiate the function

$$s(t) = t^8 + 6t^7 - 18t^2 + 2t$$

$$s'(t) = \frac{d}{dt}\left(t^8 + 6t^7 - 18t^2 + 2t\right)$$

NOTICE t INSTEAD OF x

$$= \frac{d}{dt}(t^8) + \frac{d}{dt}(6t^7) - \frac{d}{dt}(18t^2) + \frac{d}{dt}(2t)$$

$$= \frac{d}{dt}(t^8) + 6\frac{d}{dt}(t^7) - 18\frac{d}{dt}(t^2) + 2\frac{d}{dt}(t)$$

$$= 8t^7 + 6(7t^6) - 18(2t^1) + 2(1)$$

$$= \boxed{8t^7 + 42t^6 - 36t + 2}$$

Derivatives of Non-Polynomials

Like 14 on p. 197

Example. (HW Exercise 14, p. 199.)

Differentiate the function

$$H(t) = \sqrt[3]{t}\,(t+2)$$

First rewrite $H(t)$:

$$H(t) = \sqrt[3]{t}\,(t+2) = t^{1/3}(t^1+2) = t^{1/3}t^1 + 2t^{1/3}$$

$$= t^{\frac{1}{3}+1} + 2t^{1/3} = t^{4/3} + 2t^{1/3}$$

Then differentiate:

$$H'(t) = \frac{d}{dt}\left(t^{4/3} + 2t^{1/3}\right)$$

$$= \frac{d}{dt}\left(t^{4/3}\right) + \frac{d}{dt}\left(2t^{1/3}\right)$$

$$= \frac{d}{dt}\left(t^{4/3}\right) + 2\,\frac{d}{dt}\left(t^{1/3}\right)$$

$$= \frac{4}{3}t^{\frac{4}{3}-1} + 2\left(\frac{1}{3}t^{\frac{1}{3}-1}\right)$$

$$= \boxed{\frac{4}{3}t^{1/3} + \frac{2}{3}t^{-2/3}}$$

$$= \frac{4}{3}\sqrt[3]{t} + \frac{2}{3}\,\frac{1}{t^{2/3}}$$

$$= \boxed{\frac{4}{3}\sqrt[3]{t} + \frac{2}{3\left(\sqrt[3]{t}\right)^2}}$$

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Differentiation Formula of the Exponential Function, $f(x) = e^x$, Without Proof

(Sketchy proof in this section. Real proof in Section 3.5.)

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

$f(x) = e^x$ is the ONLY FUNCTION that is the same as its derivative

Derivative of x^e vs. Derivative of e^x

ex^{e-1} $\qquad$ e^x $\quad$ ~~Not xe^{x-1}~~

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Example. (HW Exercise 6, p. 199.)

Differentiate the function

$$y = 5e^x + 3$$

$$y' = \frac{d}{dx}(5e^x + 3) = \frac{d}{dx}(5e^x) + \frac{d}{dx}(3)$$

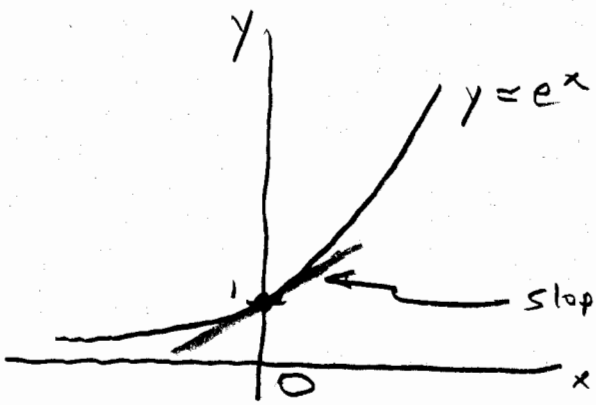
$$= 5\frac{d}{dx}(e^x) + \frac{d}{dx}(3) = 5e^x + 0 = \boxed{5e^x}$$

2. If $f(x) = e^x$, what is $f'(0)$?

$$f'(0) = \frac{d}{dx}(e^x)\Big|_{x=0} = e^x\Big|_{x=0} = e^0 = \boxed{1}$$

new notation

plug $x=0$ into e^x AFTER YOU DIFFERENTIATE



slope of tangent line at $x=0$ is 1

OR

1st $f'(x) = e^x$

2nd Plug into $x=0$: $f'(0) = 1$

What $f'(x)$ and $f''(x)$ Mean Graphically

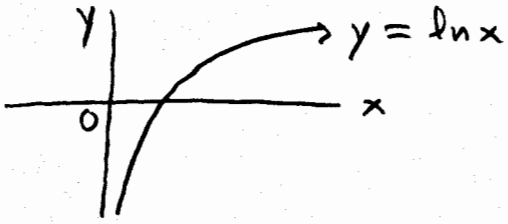
(SEE Section 2.10)

I lied - I am going to discuss a little of 2.10 now

First some definitions.

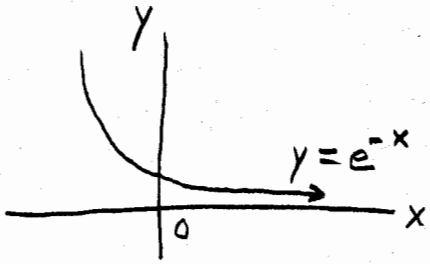
$f(x)$ is said to <u>increase</u> (over an interval) if its graph rises from left to right (over that interval)

E.g.,



$f(x)$ is said to <u>decrease</u> (over an interval) if its graph falls from left to right (over that interval)

E.g.,



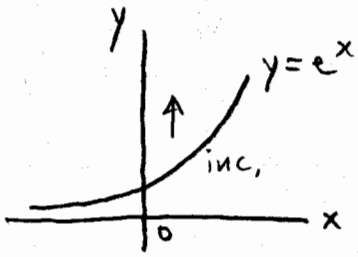
The bending of a curve is called its <u>concavity</u>:

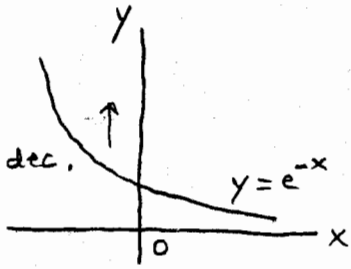
bend upward = chord above curve

bend downward = chord below curve

$f(x)$ is said to be <u>concave up</u> (over an interval) if its graph bends upward (over that interval)

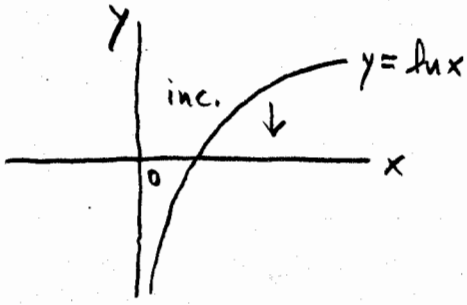
E.g.,

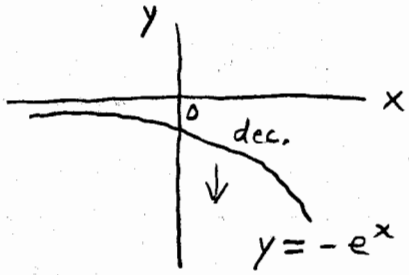




$f(x)$ is said to be <u>concave down</u> (over an interval) if its graph bends downward (over that interval)

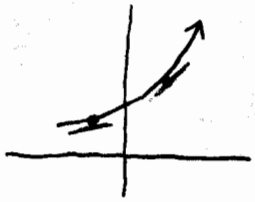
E.g.,

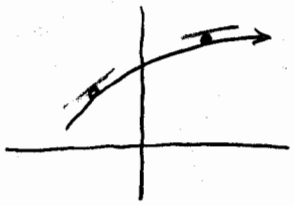




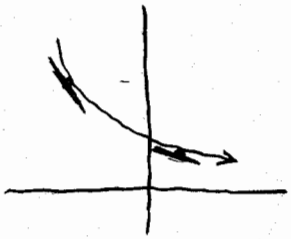
Now some concepts.

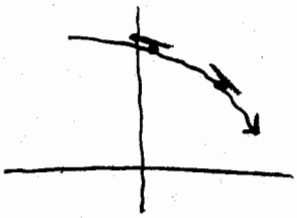
$f(x)$ increasing $\Rightarrow$ slopes > 0 $\Rightarrow$ $f'(x) > 0$



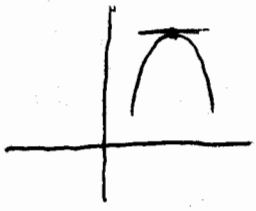


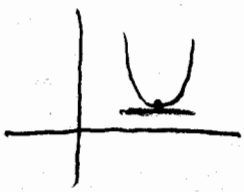
$f(x)$ decreasing $\Rightarrow$ slopes < 0 $\Rightarrow$ $f'(x) < 0$





$f(x)$ neither increasing nor decreasing $\Rightarrow$ slope $= 0$
$\Rightarrow$ $f'(x) = 0$





Leave in?

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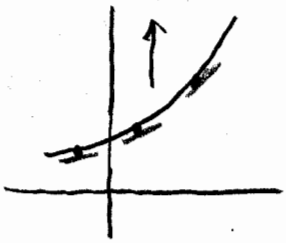
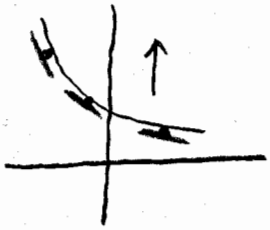
$f'(x)$ increasing $\Rightarrow$ $f''(x) > 0$ and $f(x)$ concave up

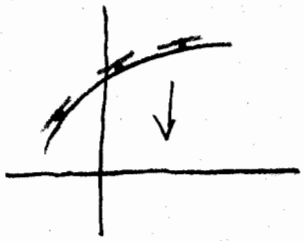


slopes getting more positive $\Rightarrow$ slopes increasing

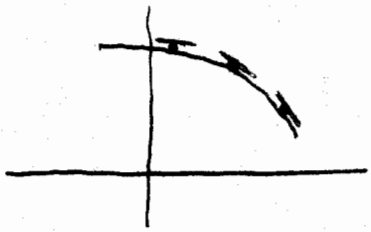


slopes getting less negative $\Rightarrow$ slopes increasing

$f'(x)$ decreasing $\Rightarrow$ $f''(x) < 0$ and $f(x)$ concave down



slopes getting less positive $\Rightarrow$ slopes decreasing



slopes getting more negative $\Rightarrow$ slopes decreasing

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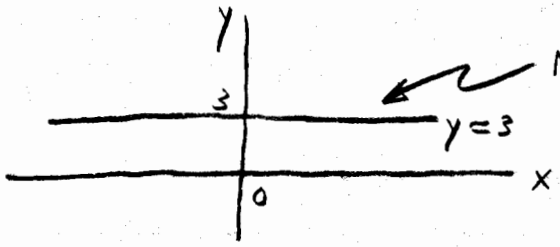
<u>Example</u>. With the constant function $f(x)=3$, graph is a horizontal line



No increasing/decreasing.
No bending upward/downward.

and $f'(x)=0$, $f''(x)=0$.

<u>Examples</u>.

1. (HW Exercise 41, p. 200.)

On what interval is the function

$$f(x) = 1 + 2e^x - 3x$$

3/20/01 Tues. Lecture increasing?

Need to find all values of x that make $f'(x) > 0$.

<u>STEP 1</u>. Find $f'(x)$.

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$$f'(x) = \frac{d}{dx}\left(1 + 2e^x - 3x\right)$$

$$= \frac{d}{dx}(1) + 2\frac{d}{dx}(e^x) - 3\frac{d}{dx}(x)$$

$$= 0 + 2e^x - 3(1)$$

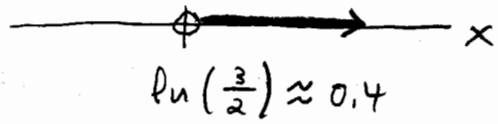
$$= 2e^x - 3$$

<u>STEP 2</u>. Set $f'(x) > 0$ and solve for x.

$$f'(x) > 0 \Rightarrow 2e^x - 3 > 0 \Rightarrow 2e^x > 3$$

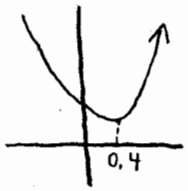
$$\Rightarrow e^x > \frac{3}{2} \Rightarrow \ln e^x > \ln\left(\frac{3}{2}\right)$$

$$\Rightarrow \boxed{x > \ln\left(\frac{3}{2}\right) \approx 0.4}$$



$$\boxed{\left(\ln\left(\frac{3}{2}\right), \infty\right)}$$

f(x) increasing over this interval



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2. (HW Exercise 42, p. 200.)

On what interval is the function

$$f(x) = x^3 - 4x^2 + 5x$$

concave up?

HINT: • Find $f'(x)$.
• Then find $f''(x)$
• Then set $f''(x) > 0$ (means "concave up") and solve for x

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3. (HW Exercise 43, p. 200.)

Find the points on the curve

$$y = x^3 - x^2 - x + 1$$

where the tangent is horizontal.

HINT: • Find $f'(x)$
• Set $f'(x) = 0$ and solve for x