3/19/01

CHAPTER 3  Differentiation

We will now learn about different RULES OF DIFFERENTIATION (= rules of finding derivatives of functions). These actually come from

1) the limit definition of derivative:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

2) the limit laws (Sect. 2.3), e.g.,

\[ \lim_{x \to 1} (5x^2 + 2x^3) = \lim_{x \to 1} 5x^2 + \lim_{x \to 1} 2x^3 \]

\[ = 5 \lim_{x \to 1} x^2 + 2 \lim_{x \to 1} x^3 \]

\[ = 5(1)^2 + 2(1)^3 \]

\[ = 7 \]

We will set aside the limit definition of derivative and start memorizing formulas and rules that come from it.
Section 3.1. Derivatives of Polynomials and Exponential Functions.

Notation: If we are given, say, the function \( f(x) = x^2 \), we will write for its derivative

\[
f'(x) = \frac{d}{dx} (x^2) \quad \text{or} \quad \frac{d}{dx} x^2 \]

---

**Power Rule**

Let \( f(x) = x \). Then

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h}
\]

\[
= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 = 1 \cdot x^0
\]
Let \( f(x) = x^2 \). Then

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.
\]

It can be shown that, in general,

\[
 \frac{d}{dx}(x^n) = nx^{n-1}, \quad n \text{ any real number!}
\]

**Power Rule**

**Examples:**

1. \( \frac{d}{dx}(x^e) = ex^{e-1} \) \( (e = 2.71828...) \)

2. \( \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2} \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}} \)

Put in Fractional Form
Recall: \( x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m \).
Example: \( x^{1/2} = \sqrt{x^1} = (\sqrt{x})^1 \).

3. \( \frac{d}{dx} \left( \frac{1}{x} \right) = \frac{d}{dx} (x^{-1}) = (-1) x^{-1-1} = -x^{-2} = \frac{-1}{x^2} \)

**Put in negative exponent form**
Recall: \( x^{-n} = \frac{1}{x^n} \).

4. \( \frac{d}{dx} (1) = \frac{d}{dx} (x^0) = 0 \cdot x^{0-1} = 0 \cdot x^{-1} = 0 \)

Recall: \( x^0 = 1 \).

5. \( \frac{d}{dx} (x) = \frac{d}{dx} (x^1) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1 \)

6. \( \frac{d}{dx} (x^2) = 2x^{2-1} = 2x^1 = 2x \)

7. \( \frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2 \)
Summary of some derivatives you may forget:

Put off #1 and 2 below

\[
\frac{d}{dx}(1) = 0 \quad \frac{d}{dx}(k) = 0, \quad k \text{ any constant}
\]

\[
E_j, \quad \frac{d}{dx}(2) = 0
\]

\[
\frac{d}{dx}(x) = 1 \quad \frac{d}{dx}(kx) = k, \quad k \text{ any constant}
\]

\[
E_j, \quad \frac{d}{dx}(3x) = 3
\]

---

**Sum/Difference and Constant Multiple Rules**

1. \[
E_j, \quad \frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) = 3x^2 + 2x
\]

2. \[
E_j, \quad \frac{d}{dx} \left( \frac{1}{x} - 1 \right) = \frac{d}{dx} \left( \frac{1}{x} \right) - \frac{d}{dx} (1)
\]

\[
= \frac{d}{dx} (x^{-1}) - \frac{d}{dx} (1)
\]

\[
= -x^{-2} - 0
\]

\[
= -\frac{1}{x^2}
\]
\[ E. \ y \ \frac{dy}{dx} \left( \frac{1}{2} \cdot x^4 \right) = \frac{1}{2} \cdot \frac{d}{dx} (x^4) = \frac{1}{2} \cdot (4x^3) = 2x^3 \]

\[ E. \ y \ \frac{dy}{dx} (2) = \frac{d}{dx} (2 \cdot 1) = 2 \cdot \frac{d}{dx} (1) = 2(0) = 0 \]

\[ E. \ y \ \frac{dy}{dx} (3x) = 3 \cdot \frac{d}{dx} (x) = 3(1) = 3 \]

Now we put these rules together to differentiate polynomial functions

Derivatives of Polynomials

Example. \ (HW Exercise 10, p. 199)

Differentiate the function

\[ \pi(t) = t^6 + 6t^2 - 18t^2 + 2t \]

\[ \pi'(t) = \frac{d}{dt} \left( t^6 + 6t^2 - 18t^2 + 2t \right) \]

Notice t instead of x
\[-180-\]

\[
\begin{align*}
\frac{d}{dt}(t^9) + \frac{d}{dt}(6t^7) - \frac{d}{dt}(18t^2) + \frac{d}{dt}(2t)
&= \frac{d}{dt}(t^9) + 6 \frac{d}{dt}(t^7) - 18 \frac{d}{dt}(t^2) + 2 \frac{d}{dt}(t) \\
&= 9t^8 + 6(7t^6) - 18(2t) + 2(1) \\
&= 9t^8 + 42t^6 - 36t + 2
\end{align*}
\]

Derivatives of Non-Polynomials

Like 14 on p. 199

Example. (HW Exercise 14, p. 199.)

Differentiate the function

\[H(t) = \sqrt[3]{t}(t + 2)\]

First rewrite \(H(t)\):

\[H(t) = \sqrt[3]{t}(t + 2) = t^{\frac{1}{3}}(t + 2) = t^{\frac{1}{3}}t + 2t^{\frac{1}{3}}
= t^{\frac{1}{3}+1} + 2t^{\frac{1}{3}} = \left(t^{\frac{4}{3}} + 2t^{\frac{1}{3}}\right)\]

Then differentiate:

\[H'(t) = \frac{d}{dt}(t^{\frac{1}{3}} + 2t^{\frac{1}{3}})\]
\[
\frac{d}{dt} \left( t^{n/2} \right) + \frac{d}{dt} \left( 2 t^{1/2} \right) \\
= \frac{4}{3} t^{3/2} + 2 \left( \frac{1}{3} t^{-1/2} \right) \\
\]

\[
= \frac{4}{3} t^{3/2} + \frac{2}{3} t^{-3/2} \\
\]

\[
= \frac{4}{3} \sqrt{t^3} + \frac{2}{3} \frac{1}{\sqrt{t}} \\
\]

\[
= \frac{4}{3} \sqrt{t^3} + \frac{2}{3} \frac{1}{(\sqrt{t})^2} \\
\]

3/20/01
Tues.
Lecture
Differentiation Formulas of the Exponential Function, $f(x) = e^x$

Without Proof

(Sketch proof in this section. Real proof in Section 3.5.)

$$\frac{d}{dx} (e^x) = e^x$$

$f(x) = e^x$ is the ONLY function that is the same as its derivative.

Derivative of $xe^x$ vs. Derivative of $e^x$

$e^{x-1} \neq xe^{x-1}$

Example. (HW Exercise 6, p.197.)

Differentiate the function

$$y = 5e^x + 3$$

$$y' = \frac{d}{dx} (5e^x + 3) = \frac{d}{dx} (5e^x) + \frac{d}{dx} (3) = 5e^x + 0 = 5e^x$$
2. If \( f(x) = e^x \), what is \( f'(0) \)?

\[
f'(0) = \left. \frac{d}{dx}(e^x) \right|_{x=0} = e^x \bigg|_{x=0} = e^0 = 1
\]

New notation:

Plug \( x = 0 \) into \( e^x \) after you differentiate.

OR

1st:
\[
f'(x) = e^x
\]

2nd:
Plug into \( x = 0 \): \( f'(0) = 1 \)

Slope of tangent line at \( x = 0 \) is 1.
What \( f'(x) \) and \( f''(x) \) Mean

**Graphically**

(SEE Section 2.10)

**First some definitions.**

\( f(x) \) is said to **increase** (over an interval) if its graph rises from left to right (over that interval).

E.g., \[ y = \ln x \]

\[ \begin{array}{c}
\text{\( y \)} \\
\hline
0 \\
x
\end{array} \]

\( f(x) \) is said to **decrease** (over an interval) if its graph falls from left to right (over that interval).

E.g., \[ y = e^{-x} \]

\[ \begin{array}{c}
\text{\( y \)} \\
\hline
0 \\
x
\end{array} \]
The bending of a curve is called its concavity:

\[ \text{bend upward} = \text{chord above curve} \]
\[ \text{bend downward} = \text{chord below curve} \]

\( f(x) \) is said to be **concave up** (over an interval) if its graph bends upward (over that interval)

\[ y = e^x \]

\[ \text{inc.} \quad \text{dec.} \]
\[ 0 \quad x \]

\( f(x) \) is said to be **concave down** (over an interval) if its graph bends downward (over that interval)

\[ y = -e^x \]

\[ \text{inc.} \quad \text{dec.} \]
\[ x \quad 0 \]
Now some concepts.

\( f(x) \) increasing \( \Rightarrow \) slopes > 0 \( \Rightarrow f'(x) > 0 \)

\( f(x) \) decreasing \( \Rightarrow \) slopes < 0 \( \Rightarrow f'(x) < 0 \)

\( f(x) \) neither increasing nor decreasing \( \Rightarrow \) slope = 0
\( \Rightarrow f'(x) = 0 \)
\( f'(x) \) increasing \( \Rightarrow \) \( f''(x) > 0 \) and \( f(x) \) concave up

slopes getting more positive \( \Rightarrow \) slopes increasing

slopes getting less negative \( \Rightarrow \) slopes increasing

\( f'(x) \) decreasing \( \Rightarrow \) \( f''(x) < 0 \) and \( f(x) \) concave down

slopes getting less positive \( \Rightarrow \) slopes decreasing

slopes getting more negative \( \Rightarrow \) slopes decreasing
Example. With the constant function \( f(x) = 3 \),
the graph is a horizontal line.

\[
\begin{array}{c}
\text{No increasing/decreasing.} \\
\text{No bending upward/downward.}
\end{array}
\]

and \( f'(x) = 0 \), \( f''(x) = 0 \).

Examples.

1. (HW Exercise 41, p. 200)

On what interval is the function

\[ f(x) = x + 2e^x - 3x \]

increasing?

**Lecture**

Need to find all values of \( x \) that make \( f'(x) > 0 \).

**STEP 1.** Find \( f'(x) \).

\[
\begin{align*}
f'(x) &= \frac{d}{dx} (x + 2e^x - 3x) \\
&= \frac{d}{dx} (1) + 2 \frac{d}{dx} (e^x) - 3 \frac{d}{dx} (x)
\end{align*}
\]

\[
\begin{align*}
&= 0 + 2e^x - 3
\end{align*}
\]

\[
\begin{align*}
&= 2e^x - 3
\end{align*}
\]

**Note:** The solution for increasing intervals of the function \( f(x) = x + 2e^x - 3x \) requires further analysis to determine the intervals where \( f'(x) > 0 \). This involves finding critical points and testing intervals to confirm the nature of the function's increase.
\[
\begin{align*}
&= 0 + 2e^x - 3(1) \\
&= 2e^x - 3
\end{align*}
\]

**STEP 2.** Set \( f'(x) > 0 \) and solve for \( x \).

\[
f'(x) > 0 \Rightarrow 2e^x - 3 > 0 \Rightarrow 2e^x > 3
\]

\[
\Rightarrow e^x > \frac{3}{2} \Rightarrow \ln e^x > \ln \left( \frac{3}{2} \right)
\]

\[
\Rightarrow x > \ln \left( \frac{3}{2} \right) \approx 0.4
\]

\[
\ln \left( \frac{3}{2} \right) \approx 0.4
\]

\[
\left( \ln \left( \frac{3}{2} \right), \infty \right)
\]

\( f(x) \) increasing over this interval
2. (HW Exercise 42, p. 200.)

On what interval is the function

\[ f(x) = x^3 - 4x^2 + 5x \]

concave up?

**HINT:**
- Find \( f'(x) \).
- Then find \( f''(x) \).
- Then set \( f''(x) > 0 \) (means "concave up") and solve for \( x \).

3. (HW Exercise 43, p. 200.)

Find the points on the curve

\[ y = x^3 - x^2 - x + 1 \]

where the tangent is horizontal.

**HINT:**
- Find \( f'(x) \).
- Set \( f'(x) = 0 \) and solve for \( x \).