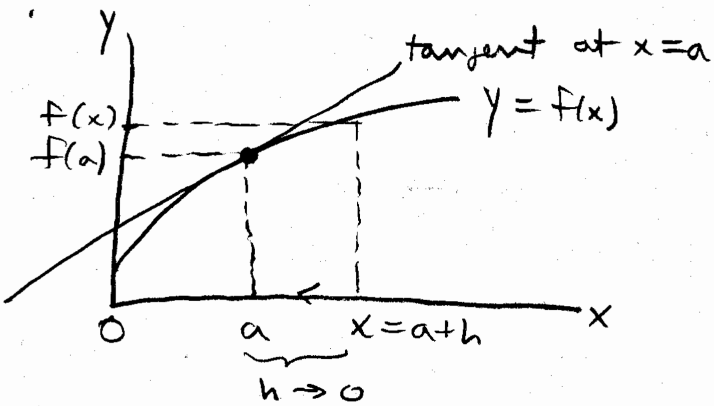
Lecture

Section 2.7. Derivatives.

Recall:



$$\left.\begin{array}{l}\text{Inst. rate of change}\\ \text{of } f \text{ at } x=a\end{array}\right\} = \left\{\begin{array}{l}\text{slope of tangent}\\ \text{line to curve } y=f(x)\\ \text{at } x=a\end{array}\right.$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{f(a+h)-f(a)}{h}$$

Defn. The instantaneous rate of change of f at $x=a$ is called the derivative of f at $x=a$ and is denoted by

$$f'(a).$$

So,

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \left\{ \begin{array}{l} \text{slope of tangent} \\ \text{line to curve} \\ y = f(x) \text{ at } x = a \end{array} \right.}$$

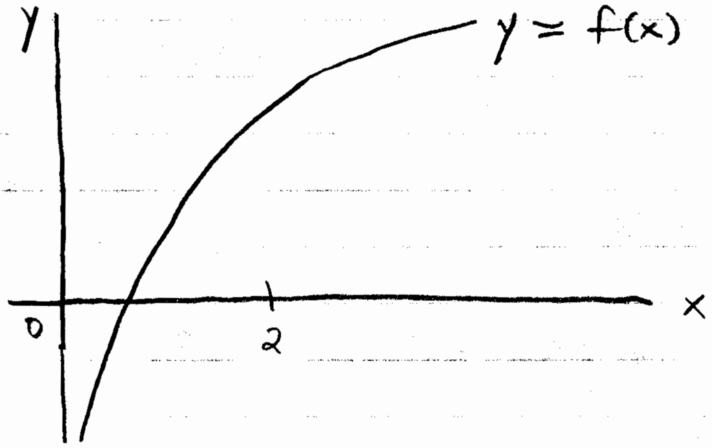
<u>Note</u>: It is convention to say that

$$\left. \begin{array}{l} \text{slope of tangent line} \\ \text{to curve } y = f(x) \text{ at} \\ x = a \end{array} \right\} = \left\{ \begin{array}{l} \text{slope of curve} \\ y = f(x) \text{ at } x = a \end{array} \right.$$

Examples. (Derivative = slope of tangent)

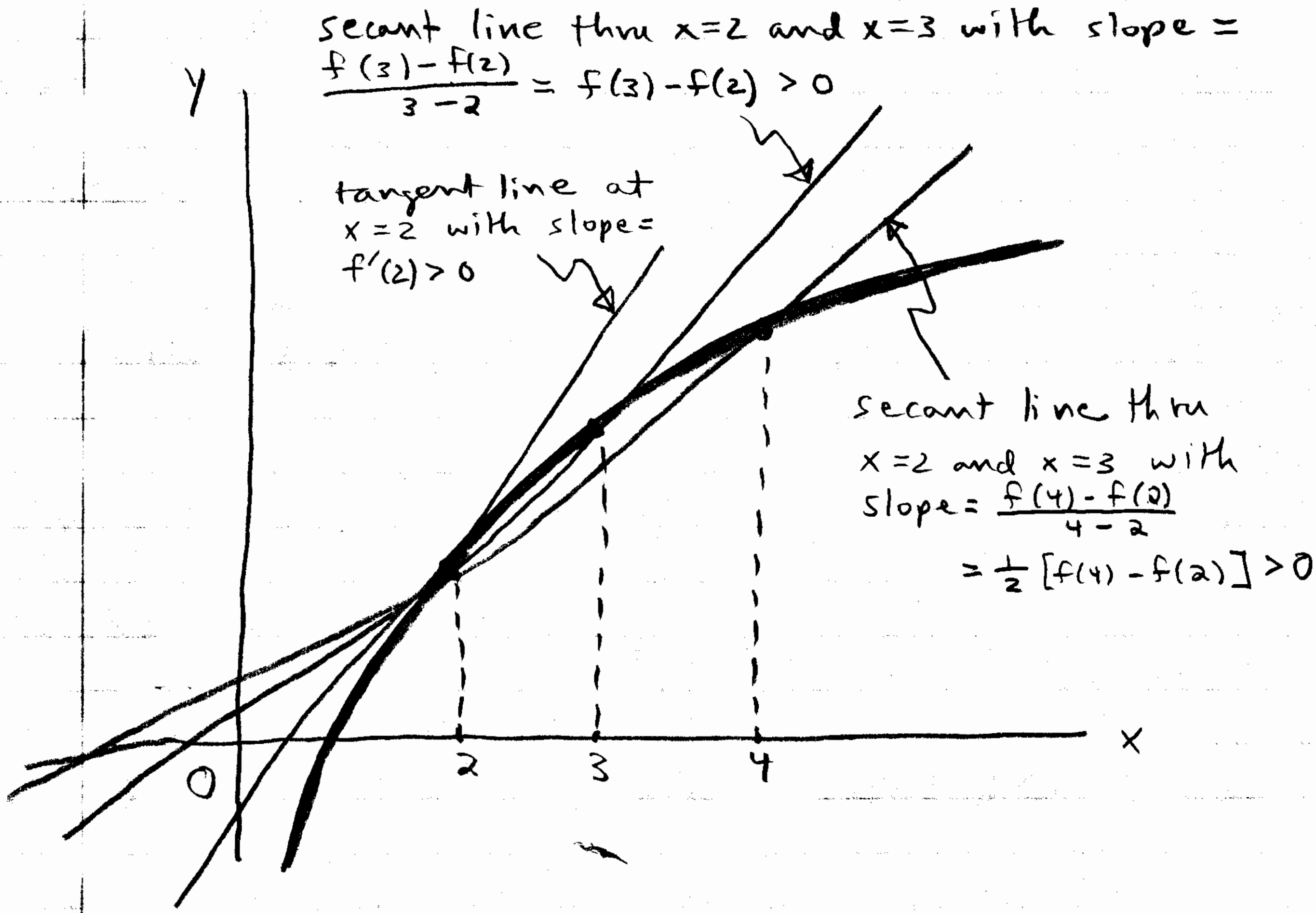
1. (HW Exercise 2, p. 156.)

For function f with graph



arrange following numbers in increasing order (smallest to largest):

$$0 \qquad f'(2) \qquad f(3) - f(2) \qquad \tfrac{1}{2}[f(4) - f(2)]$$



$$\boxed{0 < \frac{1}{2}\left[f(4)-f(2)\right] < f(3)-f(2) < f'(2)}$$

2. (HW Exercise 8, p. 156.)

If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find equation of tangent line to curve $y = 1 - x^3$ at point $(0, 1)$ $(x = 0)$.

$$g'(0) = m_{tan} = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

slope of tangent at $x = 0$

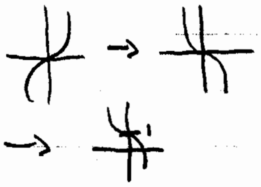
$$= \lim_{h \to 0} \frac{(\cancel{1} - h^3) - (\cancel{1} - (0)^3)}{h} = \lim_{h \to 0} \frac{-h^3}{h}$$

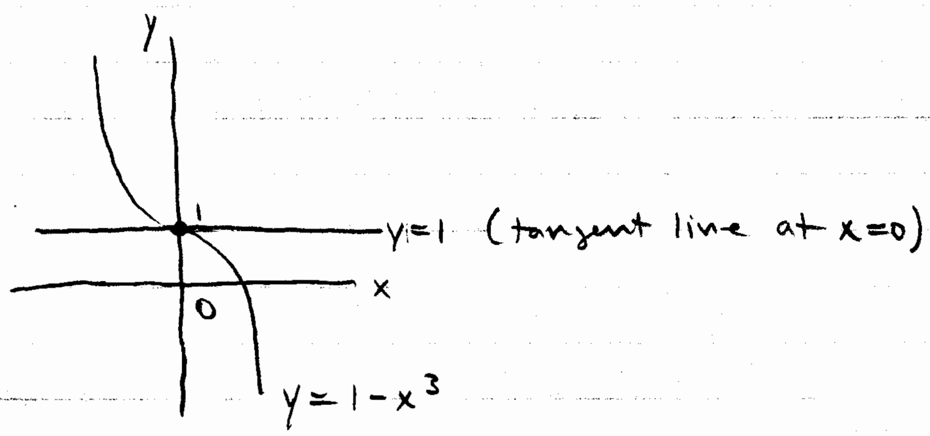
$$= \lim_{h \to 0} -h^2 = -(0)^2 = 0$$

$\therefore \quad y - y_1 = m_{tan}(x - x_1) \Rightarrow y - 1 = 0 \cdot (x - 0) \Rightarrow$

$(0, 1)$

$y - 1 = 0 \Rightarrow \boxed{y = 1}$





<u>Example.</u> (Finding f from the formula for f′)

(HW Exercise 18, p. 156.)

The limit $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$ represents the derivative of some function f at some point a. State f and a.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^3-8}{h}$$

$\therefore f(a+h) = (2+h)^3 \Rightarrow \boxed{a=2}$

$\Rightarrow f(a+h) = f(2+h)$

$= (2+h)^3$

$\Rightarrow \boxed{f(x) = x^3}$

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$f(a) = 8$: $f(a) = f(2) = 2^3 = 8$ ✓

<u>Example</u>. (Derivative = rate of change of $f(x)$ with respect to x)

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(HW Exercise 25, p. 157.)

Cost of producing x <u>ounces</u> of gold from a new gold mine is $C = f(x)$ <u>dollars</u>.

(a) What is the meaning of $f'(x)$? What are its units?

(b) What does $f'(800) = 17$ mean?

(c) SKIP

(a) $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\text{change in cost}}{\text{change in \# ounces produced}}$$

$$= \lim_{x \to a} \frac{\Delta C}{\Delta x}$$

$$= \underbrace{\frac{\Delta C}{\Delta x}}_{\substack{\text{rate of change of cost } C = f(x) \text{ with} \\ \text{respect to \# ounces } x}} \quad \text{just after (or before) } x \text{ is a}$$

$x = a$ if $f'(a)$

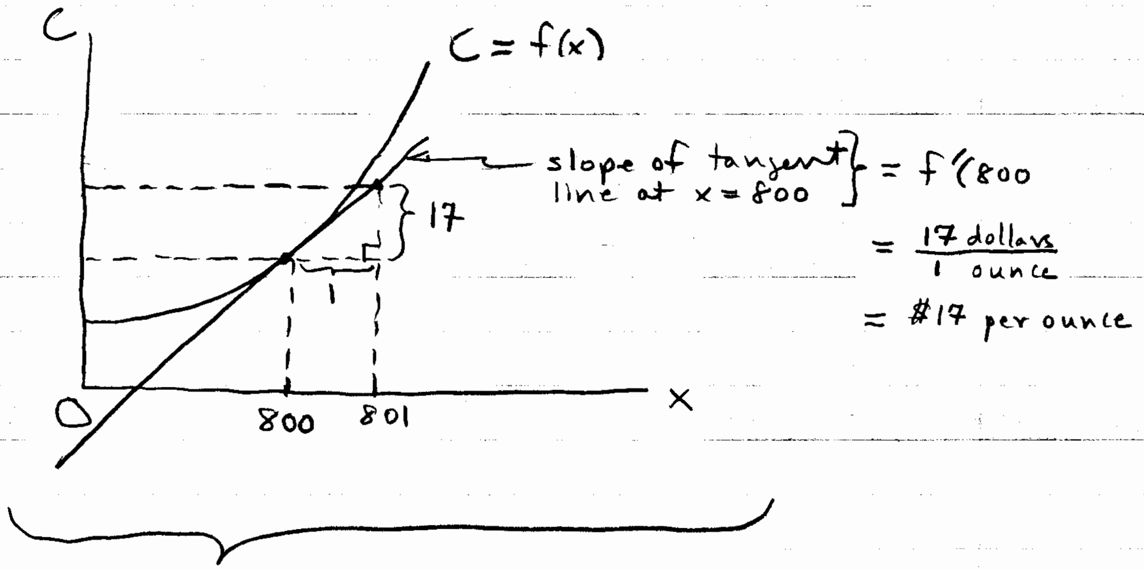
$\therefore f'(x)$ means: After x ounces of gold have been produced, the rate of change of cost with respect to the # of ounces of gold produced is $f'(x)$.

$\therefore$ Units of $f'(x)$ = Units for $\dfrac{\Delta C}{\Delta x} = \boxed{\dfrac{\text{dollars}}{\text{ounce}}}$

(b) $f'(800) = 17$ means: After 800 ounces of gold have been produced, the rate at which cost is increasing is \$17 per ounce.

the change of cost is an increase of cost since $f'(800) = +17$

($f'(800) = -17$ means: After 800 ounces produced, rate at which cost decreasing is \$17 per ounce.)



∴ Cost to produce 801^{st} ounce of gold is ~ $17 (= f'(800))

In this context, f'(800) is called the

MARGINAL COST.

Example. (Can approximate derivative from table of values of $f(x)$ versus x)

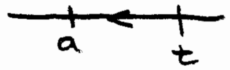
(HW Exercise 30, p. 157.)

t = time
$E(t)$ = life expectancy at birth (years) of male born in the year t in the U.S.

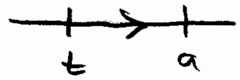
t	$E(t)$
1900	48.3
1910	51.1
1920	55.2
1930	57.4
1940	62.5
1950	65.6
1960	66.6
⋮	⋮
1990	71.8

Interpret and estimate $E'(1910)$ and $E'(1950)$

$$E'(t) = \lim_{t \to a} \frac{E(t) - E(a)}{t - a},$$ where can have $t > a$



or $t < a$



$$\approx \frac{E(t) - E(a)}{t - a}$$

$E'(1910)$:

$$t = 1920 \Rightarrow E'(1910) \approx \frac{E(1920) - E(1910)}{1920 - 1910} = \frac{55.2 - 51.1}{10} = 0.41$$

$$t = 1900 \Rightarrow E'(1910) \approx \frac{E(1900) - E(1910)}{1900 - 1910} = \frac{48.3 - 51.1}{-10} = 0.28$$

Take average of these and get even better estimate of $E'(1910)$:

$$E'(1910) \approx \frac{1}{2}(0.41 + 0.28) = \boxed{0.345}$$

$E'(1910) \approx 0,345$ means:

$$\boxed{\text{After 1910, rate at which life expectancy at birth was increasing was about 0.345 years per year.}}$$