Section 1.6. Inverse Functions and Logarithms.

Two functions are "inverses" of each other if either one "undoes" what the other is doing to $x$.

E.g., $f(x) = x^3$, $g(x) = \sqrt[3]{x}$

$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$
$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$

Notation: Inverse of $f(x)$ is $f^{-1}(x)$, and vice versa.

Warning: $f^{-1}(x) \neq \frac{1}{f(x)} = \text{reciprocal of } f(x)$

(But $(f(x))^{-1} = \frac{1}{f(x)}$)

Terminology: $f(x)$ is said to be invertible if it has an inverse.
3 Ways to Represent Inverses of Functions

1. Algebraically:

- \( f(x) \) and \( f^{-1}(x) \) are inverses if:
  1. \( f(f^{-1}(x)) = x \)
  2. \( f^{-1}(f(x)) = x \)

E.g., \( f(x) = x + 1 \) \( g(x) = x - 1 \)

\[
\begin{align*}
  f(g(x)) &= f(x - 1) = (x - 1) + 1 = x \quad \checkmark \\
  g(f(x)) &= g(x + 1) = (x + 1) - 1 = x \quad \checkmark
\end{align*}
\]

\( g(x) = f^{-1}(x) \).

2. In Tabular Form:

E.g.,

\[
\begin{array}{c|c|c|c}
\text{x} & \text{f(x)} & \text{x}^2 \\
\hline
1 & 1 & 1 \\
2 & 8 & 4 \\
3 & 27 & 9 \\
4 & 64 & 16 \\
\end{array}
\]

Switch Domain and Range of \( f \) to get \( f^{-1} \)

\[
\begin{array}{c|c|c|c}
\text{x} & \text{f^{-1}(x)} & \sqrt{x} \\
\hline
1 & 1 & 1 \\
2 & 8 & 4 \\
3 & 27 & 9 \\
4 & 64 & 16 \\
\end{array}
\]

Domain \( f \) = Range \( f^{-1} \)
Range \( f \) = Domain \( f^{-1} \)
3. **GRAPHICALLY:**

\( f(x) \) and \( f^{-1}(x) \) are inverses if the following is true:

If \((a, b)\) is a point on the graph of \( y = f(x) \) (so \( b = f(a) \)), then \((b, a)\) is a point on the graph of \( y = f^{-1}(x) \) (so \( a = f(b) \)).

This can only happen if the graph of \( y = f^{-1}(x) \) is a reflection of the graph of \( y = f(x) \) about the diagonal line \( y = x \).

E.g.,

\[
\begin{align*}
y &= x^5 \\
(2, 32) \quad y &= x \\
(32, 2) \quad y &= 5x^2
\end{align*}
\]
How to Derive \( f^{-1}(x) \) From \( f(x) \)

E.g., \( f(x) = \frac{1}{x+1} \)

STEP 1. Replace \( f(x) \) by \( y \):

\[
y = \frac{1}{x+1}
\]

STEP 2. Switch \( x \)'s and \( y \)'s:

\[
x = \frac{1}{y+1}
\]

STEP 3. Solve for \( y \):

\[
x = \frac{1}{y+1} \Rightarrow x(y+1) = 1 \Rightarrow xy + x = 1 \Rightarrow xy = 1 - x \Rightarrow y = \frac{1-x}{x} (= \frac{1}{x} - 1)
\]

STEP 4. Replace \( y \) by \( f^{-1}(x) \):

\[
f^{-1}(x) = \frac{1-x}{x}
\]
Graphical Test for Invertibility

Test to see if graph belongs to a function:

**VERTICAL LINE TEST**

E.g. $y = e^x$

- Of a function
  - Only one $y$ for each $x$
- Not of a function
  - As many as $2$ $y$'s for each $x$

Test to see if graph belongs to an invertible function:

**HORIZONTAL LINE TEST**

- Of an invertible function
  - Only one $x$ for each $y$
- Not of an invertible function
  - As many as $3$ $x$'s for each $y$
The Logarithmic Function

The logarithmic function is the inverse of the exponential function. Let us try to derive a formula for the inverse of \( f(x) = a^x \):

**STEP 1.** Replace \( f(x) \) by \( y \):

\[ y = a^x \]

**STEP 2.** Switch \( x \)'s and \( y \)'s:

\[ x = a^y \]

**STEP 3.** Solve for \( y \):

\[ x = a^y \Rightarrow \text{CANNOT solve for } y \]

\[ \text{using operations of addition/subtraction and multiplication/division} \]

\[ \checkmark \]

**INSTEAD:** We stop at \text{STEP 2} and do not solve for \( y \).
Definition. The logarithm to base \( a \) of \( x \), written 
\[
\log_a x
\]
is the power to which \( a \) must be raised to get \( x \), i.e.,
\[
\log_a x = y \quad \text{means} \quad a^y = x
\]

So, if \( f(x) = a^x \), we say that 
\[
f^{-1}(x) = \log_a x.
\]

Warning: \( y = \log_a x \) is ONLY defined for \( x > 0 \! \).

Example. Compute \( \log_3 27 \).

1st Set \( \log_3 27 = y \)

2nd \( \log_3 27 = y \) means \( 3^y = 27 \)
3rd Guess that $y$ has to be 3

\[ \log_3 27 = 3 \]
Two Important Log Functions
(On Your Calculators)

\[ \log x = \text{COMMON LOG} \]
- to the base 10:
\[ \log = \log_{10} \]
- inverse of \( \log_{10} x \) is \( 10^x \)

\[ \ln x = \text{NATURAL LOG} \]
- to the base \( e \):
\[ \ln = \log_e \]
- inverse of \( \log_e x \) is \( e^x \)

Formula to convert \( \log_a x \) to \( \ln \):
\[ \log_a x = \frac{\ln x}{\ln a} \]
(You can plug this into calculator)
## Some Important Rules of Logs and Exponents

<table>
<thead>
<tr>
<th>Exponents</th>
<th>Logs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $a^x a^y = a^{x+y}$</td>
<td>1. $\log_a xy = \log_a x + \log_a y$</td>
</tr>
<tr>
<td>2. $(a^x)^r = a^{xr}$</td>
<td>2. $r \log_a x = \log_a x^r$</td>
</tr>
<tr>
<td>3. $a^{\log_a x} = x$</td>
<td>3. $\log_a a^x = x$</td>
</tr>
<tr>
<td>4. $a^0 = 1$</td>
<td>4. $\log_a 1 = 0$</td>
</tr>
</tbody>
</table>
Graphs of Log Functions

\[ f(x) = \log_a x, \quad a > 1 \]

Eq. \( f(x) = \log x, \quad f(x) = \ln x \)

Features:
- Domain = \( \{ x \mid x > 0 \} \)
- Range = all reals
- \( x \)-intercept = 1
- Vertical asymptote: \( y\)-axis \( (x = 0) \)
- Increasing (but very gradually)
- Concave down
Will OMIT graph of \( f(x) = \log_a x \) when \( 0 < a < 1 \).