Lecture

## Section 1.6. Inverse Functions and Logarithms.

Two functions are "inverses" of each other if either one "undoes" what the other is doing to $x$.

E.g., $f(x) = x^3$ , $g(x) = \sqrt[3]{x}$

$$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$
$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

Notation: Inverse of $f(x)$ is $f^{-1}(x)$, and vice versa.

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Warning: $f^{-1}(x) \neq \frac{1}{f(x)}$ = reciprocal of $f(x)$

$$\left(\text{But } (f(x))^{-1} = \frac{1}{f(x)}\right)$$

Terminology: $f(x)$ is said to be invertible if it has an inverse.

# 3 Ways to Represent Inverses of Functions

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## 1. ALGEBRAICALLY:

$f(x)$ and $f^{-1}(x)$ are inverses if

(1) $f(f^{-1}(x)) = x$

(2) $f^{-1}(f(x)) = x$

E.g., $f(x) = x+1$ $\quad g(x) = x-1$

$f(g(x)) = f(x-1) = (x-1)+1 = x$ ✓

$g(f(x)) = g(x+1) = (x+1)-1 = x$ ✓

$\therefore \; g(x) = f^{-1}(x)$.

## 2. IN TABULAR FORM:

E.g.,

| $x$ | $f(x) = x^3$ |
|---|---|
| 1 | 1 |
| 2 | 8 |
| 3 | 27 |
| 4 | 64 |

DOMAIN RANGE

Switch DOMAIN and RANGE of $f$ to give $f^{-1}$

| $x$ | $f^{-1}(x) = \sqrt[3]{x}$ |
|---|---|
| 1 | 1 |
| 8 | 2 |
| 27 | 3 |
| 64 | 4 |

DOMAIN RANGE

Domain $f$ = Range $f^{-1}$

Range $f$ = Domain $f^{-1}$

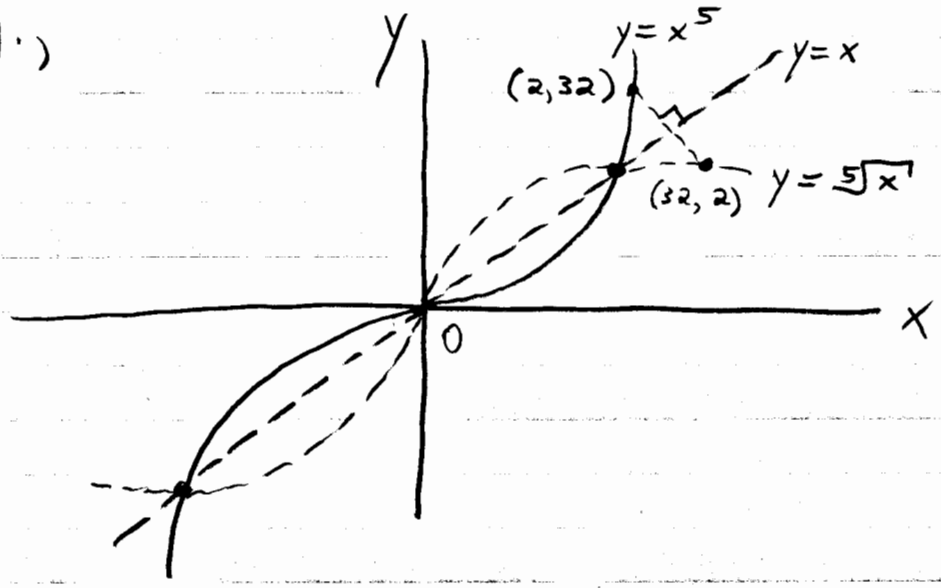
3. GRAPHICALLY:

$f(x)$ and $f^{-1}(x)$ are inverses if the following is true:

> If $(a, b)$ is a point on the graph of $y = f(x)$ (so $b = f(a)$), then $(b, a)$ is a point on the graph of $y = f^{-1}(x)$ (so $a = f(b)$).

This can only happen if the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ about the <u>diagonal line $y = x$</u>.

E.g.,



## How to Derive $f^{-1}(x)$ From $f(x)$

E.g., $f(x) = \frac{1}{x+1}$

STEP 1. Replace $f(x)$ by $y$:

$$y = \frac{1}{x+1}$$

STEP 2. Switch $x$'s and $y$'s:

$$x = \frac{1}{y+1}$$

STEP 3. Solve for $y$:

$$x = \frac{1}{y+1} \Rightarrow x(y+1) = 1 \Rightarrow xy + x = 1 \Rightarrow$$
$$xy = 1 - x \Rightarrow y = \frac{1-x}{x} \left( = \frac{1}{x} - 1 \right)$$

STEP 4. Replace $y$ by $f^{-1}(x)$:

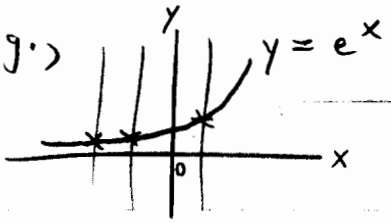
$$\boxed{f^{-1}(x) = \frac{1-x}{x}}$$

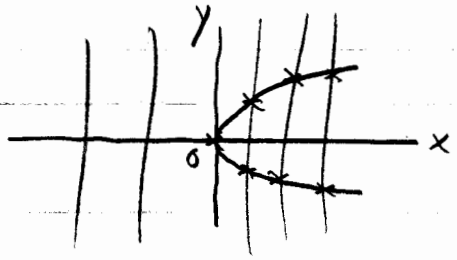
# Graphical Test for Invertibility

Test to see if graph belongs to a function:

VERTICAL LINE TEST

E.g.,




Of a function

Only one y for each x

Not of a function

As many as 2 y's for each x

Test to see if graph belongs to an invertible function:

HORIZONTAL LINE TEST

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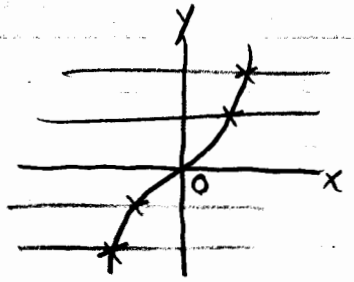


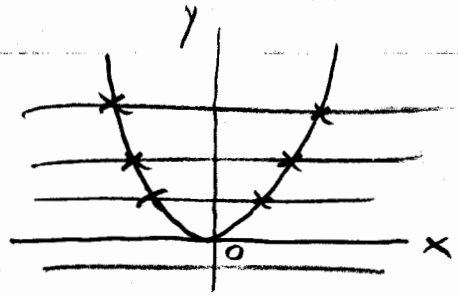


Of an invertible function

Only one x for each y

Not of an invertible function

As many as 2 x's for each y

## The Logarithmic Function

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The logarithmic function is the inverse of the exponential function.

Let us try to derive a formula for the inverse of $f(x) = a^x$:

STEP 1. Replace $f(x)$ by $y$:

$$y = a^x$$

STEP 2. Switch $x$'s and $y$'s:

$$x = a^y$$

STEP 3. Solve for $y$:

$x = a^y \Longrightarrow$ CANNOT solve for $y$ using operations of addition/subtraction and multiplication/division

INSTEAD: We stop at STEP 2 and do not solve for $y$

<u>Definition</u>. The logarithm to base $a$ of $x$, written

$$\log_a x,$$

is the power to which $a$ must be raised to get $x$, i.e.,

$$\log_a x = y \quad \text{means} \quad a^y = x$$

So, if $f(x) = a^x$, we say that

$$f^{-1}(x) = \log_a x.$$

<u>Warning</u>: $y = \log_a x$ is ONLY defined for $x > 0$!

$$\cancel{\log_a 0} \qquad \cancel{\log_a (-1)}$$

<u>Example</u>. Compute $\log_3 27$.

1st Set $\log_3 27 = y$

2nd $\log_3 27 = y$ means $3^y = 27$

$3^{\underline{rd}}$ Guess that $y$ has to be 3

$\therefore \boxed{\log_3 27 = 3}$

## Two Important Log Functions (On Your Calculators)

$\log x$ = COMMON LOG
– to the base 10 ;

$$\log = \log_{10}$$

– inverse of $\log_{10} x$ is $10^x$

$\ln x$ = NATURAL LOG
– to the base e ;

$$\ln = \log_e$$

– inverse of $\log_e x$ is $e^x$

Formula to convert $\log_a$ to $\ln$ ;

$$\log_a x = \frac{\ln x}{\ln a}$$

Can plug this into calculator

# Some Important Rules of Logs and Exponents

| | Exponents | | Logs |
|---|---|---|---|
| 1. | $a^x a^y = a^{x+y}$ | $\Rightarrow$ 1. | $\log_a xy = \log_a x + \log_a y$ |
| 2. | $(a^x)^r = a^{xr}$ | $\Rightarrow$ 2. | $r \log_a x = \log_a x^r$ |
| 3. | $a^{\log_a x} = a$ | $\Rightarrow$ 3. | $\log_a a^x = x$ |
| 4. | $a^0 = 1$ | $\Rightarrow$ 4. | $\log_a 1 = 0$ |

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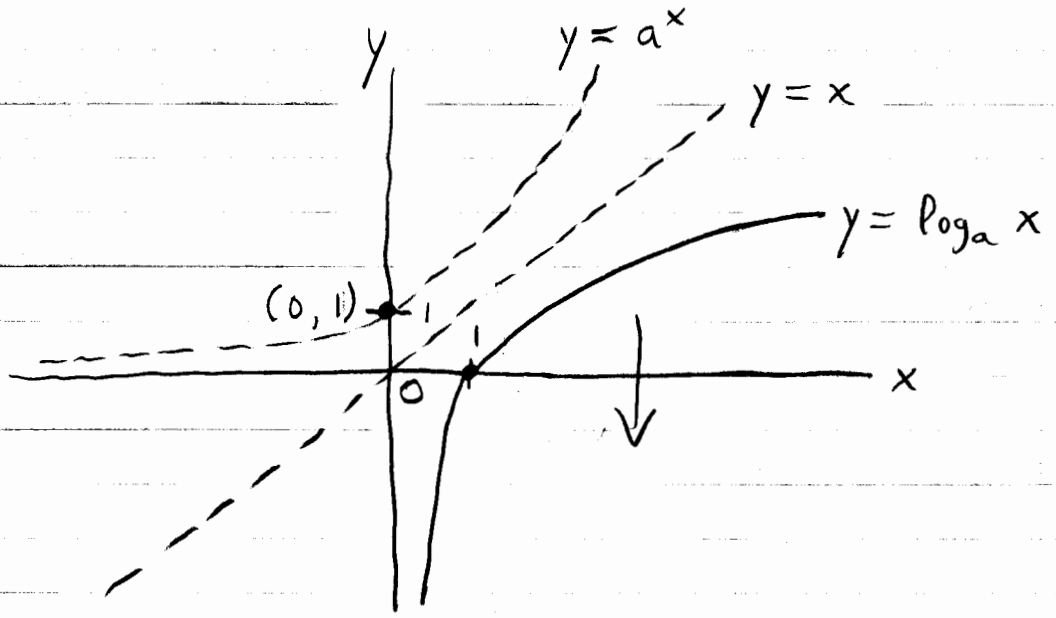
## Graphs of Log Functions

$$\boxed{f(x) = \log_a x, \quad a > 1}$$

E.g., $f(x) = \log x$, $f(x) = \ln x$




Features:
- Domain = $\{x \mid x > 0\}$
  Range = all reals
- x-intercept = 1
- vertical asymptote:
  y-axis $(x = 0)$
- increasing (BUT VERY GRADUALLY)
- concave down

[Will OMIT graph of $f(x) = \log_a x$ when $0 < a < 1$.]