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On total domination and support vertices of a tree

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Abstract

The total domination number $\gamma_t(G)$ of a simple, undirected graph G is the order of a smallest subset D of the vertices of G such that each vertex of G is adjacent to some vertex in D. In this paper we prove two new upper bounds on the total domination number of a tree related to particular support vertices (vertices adjacent to leaves) of the tree. One of these bounds improves a 2004 result of Chellali and Haynes [1]. In addition, we prove some bounds on the total domination ratio of trees.

Keywords: Total dominating set, total domination number, total domination ratio, vertex cover, trees, support vertices, leaves, isolated vertices.

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1. Definitions and introduction

A subset D of the vertices of a graph G is a *total dominating set* if every vertex of the graph is adjacent to a vertex of D (this concept was introduced by Cocknaye et al. in [3]). The *total domination number* of a graph G is the order of a smallest total dominating set, which we denote by $\gamma_t(G)$. The *total domination ratio* of G is the ratio of the total domination number to the order of G. Let T be a tree. A *leaf* of T is a vertex of degree 1 and a *support vertex* of T is a vertex adjacent to a leaf. By the *isolated support vertices* of T we mean the isolates (vertices of degree 0) of the subgraph induced by the support vertices of T. A tree is said to be *pruned* if every support vertex is adjacent to exactly one leaf. Let X be a subset of the vertices of G. The *neighborhood* of X, denoted $N(X)$, is the set of all vertices adjacent to some vertex in X. The *subgraph induced by* X is denoted $G[X]$.

The *independence number* of a graph G, denoted $\alpha(G)$, is the order of a largest subset of vertices in which no two are adjacent. A *vertex cover* of a graph G is a subset of the

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vertices that contains at least one endpoint of every edge of G. The *vertex cover number* of G, denoted $\beta(G)$, is the order of a smallest vertex cover. It is well known that for any n-vertex graph G, $\beta(G) = n - \alpha(G)$.

We prove two new upper bounds on the total domination number of a tree in this paper, one of which improves on a 2004 result of Chellali and Haynes [1]. Their result states that the total domination number of an n-vertex tree is at most $(n + |S|)/2$, where S is the set of support vertices of the tree. Our first main result improves on this inequality since, as it turns out, one may not need all support vertices and the tree should be pruned before using their bound. The statement of our first main result follows.

Theorem 1. *Let T be a non-star n-vertex tree. Then*

$$\gamma_t(T) \leq \frac{n + |S^*(T)|}{2} - \frac{|L(T)| - |S(T)|}{2},$$

where $L(T)$ is the set of leaves of T, and $S(T)$ and $S^*(T)$ are the sets of support vertices and isolated support vertices of T, respectively.

It is easily seen that the domination number of an n-vertex graph G is bounded above by $\beta(G)$, the vertex cover number of G, and that this upper bound does not hold for the total domination number. However, for V_c a smallest vertex cover of G, $G[V_c]$ the subgraph induced by V_c, and $\beta^*(G)$ the number of isolates in $G[V_c]$, we show the simple result that $\gamma_t(G) \leq \beta(G) + \beta^*(G)$ for any graph G with no isolated vertices. Now, for a tree T, our second main result improves on the latter upper bound by showing that one may not need all of the isolates in $T[V_c]$. The statement of our second main result is as follows.

Theorem 2. *Let T be an n-vertex tree with $n \geq 3$. Then*

$$\beta(T) - (k - 1) \leq \gamma_t(T) \leq \beta(T) + |S^*(T)|,$$

where $\beta(T)$ is the vertex cover number of T, $S^*(T)$ is the set of isolated support vertices of T and k is the minimum number of components taken over all subgraphs induced by minimum total dominating sets.

The proofs of these bounds are deferred to the next section of this paper. Theorem 2 provides a sufficient condition for $\gamma_t(T) = \beta(T)$.

Corollary 3. *Let T be an n-vertex tree. If T has a minimum total dominating set that induces a connected subgraph and $S^*(T) = 0$, then $\gamma_t(T) = \beta(T)$.*

Note that since $\beta(G) = n - \alpha(G)$, both of the new upper bounds involve the order of the graph and thus both suggest corollaries that provide sufficient conditions for when the total domination number of a tree is at most half its order. In the last section of this paper we note those conditions and present another. We also prove a general lower bound on the total domination ratio of trees.

Lastly, note that the upper bound of Theorem 2 was conjectured by Graffiti.pc, a conjecture-making computer program written by E. DeLaViña. The operation of Graffiti.pc and its complete list of conjectures can be found in [5] and [6], respectively. The upper bound of Theorem 1 is proven as an application of another of Graffiti.pc's conjectures, which is presented and proved first in the next section.

2. Proofs of bounds on $\gamma_t(T)$

The bound proven next originated as Graffiti.pc's conjecture number 332 in [6]. The proof of Theorem 1 will follow as an application of this result. Before proving the theorem, we present some definitions needed for the proof. A *branch point* of a tree is a vertex of degree at least three. Each leaf of a non-path tree has an associated nearest branch point. For any leaf p in a non-path tree T, let $d(p)$ be the distance from p to the nearest branch point v, and $p = v_1, v_2, \ldots v_{d(p)}, v$ be the unique path from p to v in T. Call the path L_p from a leaf p to $v_{d(p)}$ (the last vertex before the nearest branch point) a *branch* of the tree. Every non-path tree then has exactly as many branches as leaves. Call a branch point v *accessible* to a leaf p if the unique path from v to p does not contain any branch points besides v. It is clear that every non-path tree contains a branch point which is accessible to at least two leaves.

Theorem 4. *Let T be a n-vertex tree with $n \geq 3$. Then*

$$\gamma_t(T) \leq \frac{n + |S^*(T)|}{2},$$

where $S^*(T)$ is the set of isolated vertices in the subgraph induced by the support vertices of T.

Proof. The truth of the statement for small trees is clear, since the relation holds for any tree with radius $r = 1$ on more than two vertices, and also for paths with more than two vertices. Assume the statement is true for trees with no more than k vertices. Let T be a tree with $k + 1$ vertices. Moreover, assume that T is not a path and that $r > 1$, since otherwise we are done. Thus, T has branch points. Let v be a branch point that is accessible to at least two leaves p and q. Let L_p and L_q be the branches associated to these leaves. So L_p is the path $p = p_1, p_2, \ldots, p_{d(p)}$, and L_q is the path $q = q_1, q_2, \ldots, q_{d(q)}$. Note that both $p_{d(p)}$ and $q_{d(q)}$ are adjacent to the branch point v.

Consider first the case where either of the branches L_p or L_q has three or more vertices, say L_p (that is, $d(p) \geq 3$). Let $T' = T - \{p, p_2, p_3\}$ (that is, the tree induced on $V(T) \setminus \{p, p_2, p_3\}$). Then $n(T) = n(T') + 3$. Since v is a branch point accessible to p, T' must be a tree with more than two vertices. Let $S^*(T)$ and $S^*(T')$ be the sets of isolated vertices in the subgraphs induced by the support vertices of T and T', respectively. By induction, $\gamma_t(T') \leq \frac{1}{2}[n(T') + |S^*(T')|]$. Let D' be a smallest total dominating set of T'. Since $D = D' \cup \{p_2, p_3\}$ is a total dominating set of T, $\gamma_t(T) \leq \gamma_t(T') + 2$. Suppose

$|S^*(T')| = |S^*(T)| - 1$. This can happen if no new isolated support vertices are introduced in T'. Thus $\gamma_t(T) \leq \gamma_t(T') + 2 \leq \frac{1}{2}[n(T') + |S^*(T')|] + 2 = \frac{1}{2}[n(T) - 3 + |S^*(T)| - 1] + 2 = \frac{1}{2}[n(T) + |S^*(T)|]$. On the other hand, if $|S^*(T)| = |S^*(T')|$, then since p_2 is not in $S^*(T')$, some vertex along the branch L_p, or perhaps v is now in $S^*(T')$. In this case, put $T' = T - \{p, p_2, p_3, p_4\}$, and observe that $|S^*(T')| \leq| S^*(T)|$. Since $D = D' \cup \{p_2, p_3\}$ is again a total dominating set of T, we have $\gamma_t(T) \leq \gamma_t(T') + 2$. Then $\gamma_t(T) \leq \gamma_t(T') + 2 \leq \frac{1}{2}[n(T') + |S^*(T')|] + 2 \leq \frac{1}{2}[n(T) - 4 + |S^*(T)|] + 2 = \frac{1}{2}[n(T) + |S^*(T)|]$.

Now, assume that each of L_p and L_q has two or fewer vertices. By symmetry it is enough to consider the cases where (1) $d(p) \leq 2$ and $d(q) = 1$ or (2) $d(p) = 2$ and $d(q) = 2$.

Case (1). Suppose $d(p) \leq 2$ and $d(q) = 1$. Let $T' = T - \{q\}$. Let D' be a minimum total dominating set for T' that contains no leaves of T'. Then it is easily seen that D' must contain v, which, in T, dominates q. Thus D' is also a total dominating set for T, and $\gamma_t(T) \leq \gamma_t(T')$. However, provided p_2 exists, it is not an isolated support vertex in T but may be such in T'. Therefore, $|S^*(T')| \leq |S^*(T)| + 1$. So, by induction $\gamma_t(T) \leq \gamma_t(T') \leq \frac{1}{2}[n(T') + |S^*(T)|] \leq \frac{1}{2}[n(T) - 1 + |S^*(T)| + 1] \leq \frac{1}{2}[n(T) + |S^*(T)|]$.

Case (2). Suppose $d(p) = 2$ and $d(q) = 2$. It can be assumed that v is not adjacent to a leaf, that is, v is not a support vertex in T. Otherwise Case (1) could be applied. Let $T' = T - \{q\}$. Let D' be a minimum total dominating set for T' that contains no leaves of T'. In particular, D' does not contain q_2. Since $D = D' \cup \{q_2\}$ is a total dominating set of T, we have $\gamma_t(T) \leq \gamma_t(T') + 1$. Since v is not a support vertex (by assumption), q_2 is an isolated support vertex in T. All isolated support vertices in T' are also isolated support vertices in T. Thus, $|S^*(T)| = |S^*(T')| + 1$. So, by induction $\gamma_t(T) \leq \gamma_t(T') + 1 \leq \frac{1}{2}[n(T') + |S^*(T')|] + 1 = \frac{1}{2}[n(T) - 1 + |S^*(T)| - 1] + 1 \leq \frac{1}{2}[n(T) + |S^*(T)|]$. □

Proof of Theorem 1. Let T' be the pruned tree formed by deleting all but one leaf from each support vertex of T. Then clearly, $n' = n - (|L(T)| -| S(T)|)$, $\gamma_t(T) = \gamma_t(T')$, $|S(T)| = |S(T')| = |L(T')|$ and $|S^*(T)| = |S^*(T')|$. Now, by Theorem 4 and the latter observations we see

$$\begin{aligned}
\gamma_t(T) &= \gamma_t(T') \leq \frac{n' + |S^*(T')|}{2} \\
&= \frac{n - (|L(T)| - |S(T)|) + |S^*(T)|}{2} \\
&= \frac{n + |S^*(T)|}{2} - \frac{|L(T)| - |S(T)|}{2}.
\end{aligned}$$

□

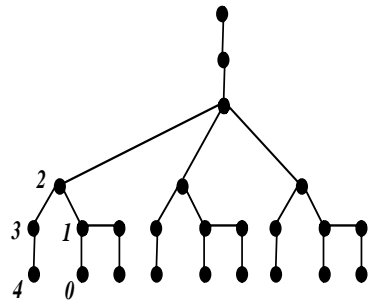

Figure 1: $\gamma_t = 14$, $n = 24$, $|S^*| = 4$ and $|L| = |S| = 10$

Label a path on 5 vertices left to right $0 - 1 - 2 - 3 - 4$ with 0 and 4 the labels of the endpoints. To see that the bound in Theorem 1 is sharp for infinitely many trees, begin by joining the endpoint of a path on 2 vertices to vertex 1 of the labeled path on 5 vertices. Call the resulting tree H. Now for $m \geq 1$, let $T(m)$ be the union of m copies of H and a path on 3 vertices, such that one endpoint of the path on 3 vertices is joined to each of the vertices labeled 2 in the m copies of H (see Figure 1 for $T(3)$). It is easily verified that for $m \geq 1$, $\gamma_t(T(m)) = 4m + 2$, $n(T(m)) = 7m + 3$, $|S^*(T(m))| = m + 1$ and $|S(T(m))| = |L(T(m))| = 3m + 1$.

Theorem 5. *Let G be an n-vertex graph with no isolated vertices such that $n \geq 2$ and let V_c be a smallest vertex cover of G. Then*

$$\gamma_t(G) \leq |V_c| + \beta^*(G),$$

where $\beta^*(G)$ is the number of isolates in $G[V_c]$.

Proof. Let $I = V(G) - V_c$. Since I is a maximum independent set, every vertex of V_c must have a neighbor in I. Let D^* be a smallest subset of vertices of I that dominate the isolated vertices of $G[V_c]$. It is easily seen that $|D^*| \leq \beta^*(G)$ and $V_c \cup D^*$ is a total dominating set. Thus, $\gamma_t(G) \leq |V_c \cup D^*| \leq| V_c| + |D^*| \leq| V_c| + \beta^*(G)$. □

The lemma below is proven as Lemma 1 in [4]. We use it to prove the lower bound of Theorem 2.

Lemma 6. *Let T be a tree with dominating set D. Then the subgraph induced by $V(T) - D$ has at most $k - 1$ edges, where k is the number of components of the subgraph induced by D.*

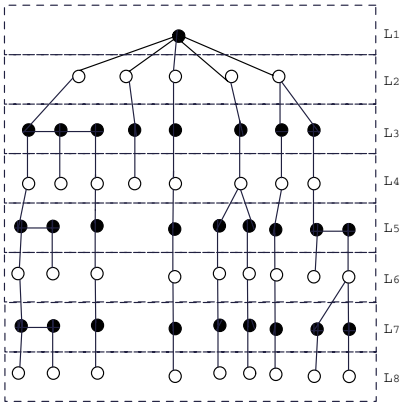

Figure 2: Partitioning of $V(T)$ in the proof of Theorem 2.

Proof of Theorem 2. Let D be a minimum total dominating set. Let D' be a smallest set of vertices in $V(T) - D$ that cover the edges of the subgraph induced by $V(T) - D$. Next put $C = D \cup D'$. Since C is clearly a vertex cover and by Lemma 1 we see $|D'| \leq k - 1$, it follows that $\beta(T) \leq |C| \leq (k-1) + \gamma_t(T)$.

To show the upper bound, let I be a maximum independent set containing the leaves of T. Then $\alpha(T) = |I|$ and $V_c = V - I$ is a vertex cover for T. Observe that by choice of I the support vertices S of T are contained in V_c. We can assume that $T[V_c]$ has at least one isolated vertex, say x^*, that is not a support vertex, otherwise the result follows from Theorem 5.

Next we will partition the vertices of T as follows. Let $L_1 = \{x^*\}$ and $L_2 = N(x^*)$ (we can visualize that the tree is rooted at x^*). Observe that $L_1 \subseteq V_c$ and $L_2 \subseteq I$. Now let

$$L_3 = [N(L_2) - L_1] \cup C_{N(L_2)-L_1},$$

where $C_{N(L_2)-L_1} = \{v \in V_c | v \text{ is in a component of } V_c \text{ with a vertex of } N(L_2) - L_1\}$. In other words, L_3 contains the neighborhood of L_2 not already placed into a block of our partition and also the vertices of all components of V_c that contain these neighbors (see Figure 2 for an illustration of the partitioning). Next we put $L_4 = N(L_3) - L_2$ and observe that $L_4 \subseteq I$, since $L_3 \subseteq V_c$.

We continue this process so that at some odd number step $2k+1$, we put

$$L_{2k+1} = [N(L_{2k}) - L_{2k-1}] \cup C_{N(L_{2k})-L_{2k-1}},$$

where $C_{N(L_{2k})-L_{2k-1}} = \{v \in V_c | v \text{ is in a component of } V_c \text{ with a vertex } N(L_{2k}) - L_{2k-1}\}$, and we put $L_{2k+2} = N(L_{2k+1}) - L_{2k}$. Since T is a finite tree, this process will terminate at some m^{th} step where m is even and composed only of leaves. This partition $\{L_1, L_2, ..., L_m\}$ of $V(T)$ clearly has the following properties.

a. $I = L_2 \cup L_4 \cup ... \cup L_{m-2} \cup L_m$.

b. $V_c = L_1 \cup L_3 \cup ... \cup L_{m-3} \cup L_{m-1}$.

c. For $i > 1$, if $v \in L_i$ has a neighbor in L_{i-1}, then it has only one neighbor in L_{i-1}.

Now let $D = V_c$. If D is a total dominating set, then the result holds since $\gamma_t(T) \leq |D| = \beta(T)$. Otherwise, we alter D as follows. If D contains isolated vertices of $T[V_c]$ that are support vertices of T, then let D^* be a smallest subset of vertices of $I - L$ that dominate these isolated support vertices. Clearly $|D^*| \leq |S^*|$. Put the vertices of D^* into D and observe that

$$|D| \leq \beta(T) + |S^*|.$$

If D is now a total dominating set, then the result holds since $\gamma_t(T) \leq |D| \leq \beta(T) + |S^*|$. In case D is not yet a total dominating set, we alter D without increasing its order as follows. In decreasing order we visit each L_i with odd index i where $3 \leq i \leq m-1$. Thus we begin with L_{m-1} and observe that if there is an isolate of $T[V_c]$ in L_{m-1} then it is a support vertex and some vertex of D^* (which we previously added to D) is adjacent to it. Now for each non-support isolate x of $T[V_c]$ that is in L_{m-3}, if all of the neighbors of x that are in L_{m-2} are dominated by $L_{m-1} \cap D$, then remove x from D and add to D its unique neighbor in L_{m-4}, otherwise we leave x in D. Continue this way for each odd i in decreasing order. That is in general for L_i where i is odd, if a non-support isolate x of $T[V_c]$ is in L_i, and all of the neighbors of x that are in L_{i+1} are dominated by $L_{i+2} \cap D$, then remove x from D and add its unique neighbor in L_{i-1} to D. Otherwise we leave x in D. This process terminates after $i = 3$. Now if some vertex of L_2 is in D then we are done, otherwise remove x^* from D and replace it with one of its neighbors. Note that in either case, x^* will be dominated by D.

It is clear that the order of D has not increased, thus, once we show that D is a total dominating set, the result follows. Since D contains all support vertices of T and contains D^*, it is clear that the support vertices and the leaves of T are dominated. Moreover, the non-isolated vertices of V_c are in D and thus also clearly dominated by D. What remains to be verified is that non-support isolates of V_c and non-leaf vertices of the independent set I are dominated by D. Let x_i be a non-support isolate of V_c and suppose that it is in L_i, where i is odd and $1 < i < m-1$. If x_i is not in D (that is it was removed), then its neighbor in L_{i-1} must be in D and thus x_i is dominated by D. On the other hand, if x_i is in D (that is it did not get removed from D), then at least one of its neighbors, say y, in L_{i+1} is not dominated by $L_{i+2} \cap D$. But the vertices of L_{i+2} are in the vertex cover and we have assumed y is not a leaf (since x_i is not a support), so y must be adjacent to a vertex of L_{i+2}. Thus, y must be adjacent to an isolate in L_{i+2} that was removed from D. But in this case y must have been placed in D and thus x_i is dominated by D. Finally, let w be a non-leaf vertex of the independent set I and suppose that it is in L_j where j is even and $2 \leq i < m$. The only neighbors of w are in V_c. If w is adjacent to a non-isolate of V_c, then it is dominated by D. Now assume w is only adjacent to isolates of V_c, and let $z \in V_c$ be its neighbor in L_{j-1}. If z is not in D, then w must be dominated by L_{j+1}, otherwise z would still be in D. $\square$

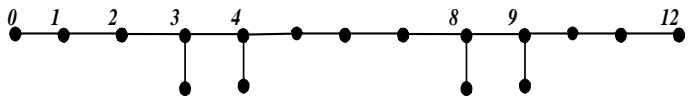

Figure 3: $\gamma_t(T_2(2)) = 9$, $n(T_2(m)) = 17$, $\alpha(T_2(m)) = 10$ and $|S^*(T_2(m))| = 2$

To see that the bound in Theorem 2 is sharp for infinitely many trees and sometimes better than the bound in Theorem 1, begin with a path P_{5m+3} on $5m+3$ vertices for $m \geq 1$ and assume that the vertices are labeled left to right $0-1-2-3-...-(5m+2)$ with 0 and $5m+2$ the labels of the endpoints. Now for each vertex of P_{5m+3} whose label is congruent to 3 modulo 5 or to 4 modulo 5 identify it with an endpoint of a path on 2 vertices. Call the resulting tree $T_2(m)$ (see Figure 3 for $T_2(2)$). It is easily verified that for $m \geq 1$, $\gamma_t(T_2(m)) = 3m+3$, $n(T_2(m)) = 7m+3$, $\beta(T_2(m)) = 3m+1$ and $|S^*(T_2(m))| = 2$.

3. The total domination ratio of a tree

It is well known that the total domination number of a connected graph is at most two-thirds its order (see [3]). It is also known that there are many trees whose total domination number is equal to two-thirds the order. In this section, we discuss new upper and lower bounds for the total domination ratio of trees. From our first two results we get as a special case that trees with no isolated support vertices have their total domination ratios bounded above by a half.

Theorem 7. [3] *For any connected n-vertex graph G with $n \geq 3$, $\gamma_t(G) \leq \frac{2}{3}n$.*

We begin with the conditions suggested by our theorems.

Corollary 8. *Let T be an n-vertex non-star tree with $L(T)$ and $S(T)$ the sets of leaves and support vertices, respectively. If $|L(T)| \geq |S(T)| + |S^*(T)|$, then $\gamma_t(T) \leq \frac{n}{2}$.*

Proof. This follows easily from Theorem 1. □

Corollary 9. *Let T be an n-vertex tree with $S^*(T)$ its set isolated vertices in the subgraph induced by support vertices. If $\alpha(T) - |S^*(T)| \geq \frac{n}{2}$, Then $\gamma_t(T) \leq \frac{n}{2}$.*

Proof. This follows easily from Theorem 2. □

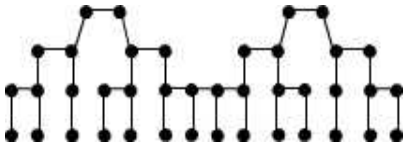

Figure 4: $n = 40$, $|N(S)| = 34$, $|S| = 14$ and $\gamma_t = 18$

Inspired by the sufficient conditions provided by Corollaries 8 and 9, Graffiti.pc was queried specifically for such conditions. The next theorem was among its conjectures. The graph in Figure 4 has total domination number less than half its order, although it does not satisfy either of the conditions presented thus far. However, it does satisfy the condition in the next theorem.

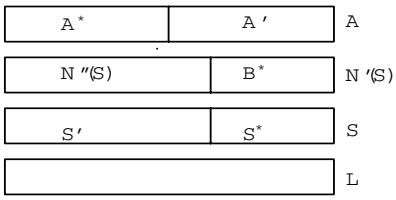

Figure 5: Partition of the vertices in the proof of Theorem 10.

Theorem 10. *Let T be an n-vertex tree with $n \geq 4$.*

$$\textit{If } |N(S)| - |S| \geq \tfrac{n}{2}, \textit{ then } \gamma_t(T) \leq \tfrac{n}{2},$$

where $S = S(T)$ is the set of support vertices of T and $N(S)$ the neighborhood of S.

Proof. Let S^* be the set of isolated vertices in the subgraph induced by the support vertices S of T, and put $S' = S - S^*$. The non-leaf and non-support neighbors of S we denote by $N'(S)$, that is $N'(S) = N(S) - (L \cup S)$. Clearly L, $N'(S)$ and S' are pairwise disjoint and $N(S) = L \cup S' \cup N'(S)$. Next, let B^* be a smallest subset of $N'(S)$ that dominates the vertices in S^*; clearly, $|B^*| \leq |S^*|$. Let $N''(S) = N'(S) - B^*$ and observe that

$$N(S) = L \cup N''(S) \cup B^* \cup S'. \tag{1}$$

Moreover, since L, $N''(S)$, B^* and S' are pairwise disjoint,

$$|N(S)| = |L| + |N''(S)| + |B^*| + |S'|. \tag{2}$$

Let $A = (V(T) - N(S)) - S^*$. Then by our partition of the vertices of T and (1) it follows that $V(T) - N(S) = A \cup S^*$, and so

$$|V(T)| - |N(S)| = |A| + |S^*|. \tag{3}$$

By assumption and (3) we see

$$\frac{n}{2} \geq |V(T)| - (|N(S)| - |S|) = |V(T)| -| N(S)| + |S| = |A| + |S^*| + |S|,$$

which yields

$$|A| + |S^*| + |S| \leq \frac{n}{2}. \tag{4}$$

Lastly, let A^* be the isolates in the subgraph induced by A, and put $A' = A - A^*$. Let D^* be the smallest subset of $N'(S)$ that dominates the vertices of A^*. Put $D = D^* \cup A' \cup B^* \cup S$. It is easily seen that D is a total dominating set. Thus, by construction of D, the observation $|D^*| \leq |A^*|$, our partition of A and (4), respectively, we see that

$$\begin{aligned} \gamma_t(T) \leq |D| &\leq |D^*| + |A'| + |B^*| + |S| \\ &\leq |A^*| + |A'| + |S^*| + |S| \\ &\leq |A| + |S^*| + |S| \\ &\leq \frac{n}{2}. \end{aligned}$$

□

Theorem 12 and its corollary provide a general lower bound on the total domination ratio of trees. Theorem 12 uses another result of Chellali and Haynes [2].

Theorem 11. [2] *Let T be an n-vertex tree with $n > 2$. Then*

$$\gamma_t(T) \geq \tfrac{n - |L(T)| + 2}{2},$$

where $L(T)$ is the set of leaves.

Theorem 12. *Let T be an n-vertex tree. Let m be a positive constant such that $m\gamma_t(T) \geq |L(T)|$. Then*

$$\tfrac{\gamma_t(T)}{n} > \tfrac{1}{m+2},$$

where $L(T)$ is the set of leaves.

Proof. Clearly, by inspection, $n > 2$. By Theorem 11, we have $2\gamma_t(T) \geq n - |L(T)| + 2 > n - |L(T)|$. Now by the choice of m,

$$\frac{1}{m+2} = \frac{\gamma_t(T)}{(m+2)\gamma_t(T)} = \frac{\gamma_t(T)}{m\gamma_t(T) + 2\gamma_t(T)} < \frac{\gamma_t(T)}{|L(T)| + n - |L(T)|} = \frac{\gamma_t(T)}{n}.$$

□

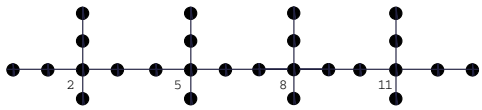

Figure 6: $\gamma_t(T_4) = 10$ and $n(T_4) = 26$

Corollary 13. *Let T be an n-vertex tree. Let m be the average number of leaves per support vertex. Then*

$$\frac{\gamma_t(T)}{n} > \frac{1}{m+2}.$$

Corollary 13, for instance, implies that the total domination ratio of a pruned tree is more than $\frac{1}{3}$. To see that a ratio of $\frac{1}{3}$ is best possible for pruned trees, begin with a path P on $3m+2$ vertices with $m \geq 3$. Label the vertices of P left to right so that 0 and $3m+1$ are the endpoints. Now, identify a non-endpoint of a path on 4 vertices with each vertex of P that is labeled $k \equiv 2 (\text{mod } 3)$. Let T_m be the resulting pruned tree; see Figure 6 for T_4. It is easy to see that $n(T_m) = 6m + 2$ and $\gamma_t(T_m) = 2m + 2$.

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