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# NOTES ON THE INDEPENDENCE NUMBER IN THE CARTESIAN PRODUCT OF GRAPHS

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**Abstract**

Every connected graph $G$ with radius $r(G)$ and independence number $\alpha(G)$ obeys $\alpha(G) \geq r(G)$. Recently the graphs for which equality holds have been classified. Here we investigate the members of this class that are Cartesian products. We show that for non-trivial graphs $G$ and $H$, $\alpha(G \Box H) = r(G \Box H)$ if and only if one factor is a complete graph on two vertices, and the other is a nontrivial complete graph. We also prove a new (polynomial computable) lower bound $\alpha(G \Box H) \geq 2r(G)r(H)$ for the independence number and we classify graphs for which equality holds.

The second part of the paper concerns independence irreducibility. It is known that every graph $G$ decomposes into a König-Egervary subgraph (where the independence number and the matching number sum to the number of vertices) and an *independence irreducible* subgraph (where every non-empty independent set $I$ has more than $|I|$ neighbors). We examine how this decomposition relates to the Cartesian product. In particular, we show that if one of $G$ or $H$ is independence irreducible, then $G \Box H$ is independence irreducible.

**Keywords:**

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## 1. Introduction

This paper explores relationships between Cartesian products of graphs and recent results on independence. It consists of an introduction followed by two independent sections. Research on the independence number of a Cartesian product of graphs has been ongoing [1, 8, 9, 11, 12, 13, 18].

Section 2 concerns those graphs whose independence number and radius are equal, a class that has recently been characterized [3]. Here we show a nontrivial Cartesian product $G\Box H$ belongs to this class if and only if both $G$ and $H$ are complete and one is $K_2$. We also prove a new polynomial time computable lower bound $\alpha(G\Box H) \geq 2r(G)r(H)$ for the independence number, and we characterize those graphs for which equality holds.

Section 3 is concerned with the notion of independence irreducibility. A graph is said to be *independence irreducible* if every non-empty independent set $I$ has more than $|I|$ neighbors. It has been shown that any graph can be decomposed into two unique subgraphs, one independence irreducible and the other König-Egervary, and that the problem of finding the independence number of any graph can be reduced in polynomial-time to the problem of finding the independence number of this independence irreducible subgraph [14, 15, 16]. The main result of Section 3 is that if one of $G$ or $H$ is independence irreducible, then so is $G\Box H$. Other questions regarding the relationships between the independence decomposition structure of the factors and that of the product are discussed. We close with a set of conjectures and open questions.

To set the stage for all of this, the remainder of this introduction recalls some fundamental definitions and ideas.

All our graphs are finite and simple. The *order* of a graph $G = (V(G), E(G))$ is $n(G) = |V(G)|$. The *eccentricity* of a vertex of a connected graph is the maximum distance from the vertex to any other vertex. The *radius* of $G$, denoted by $r(G)$, is the minimum eccentricity of the vertices of the graph. We denote by $G\langle S\rangle$ the subgraph of $G$ induced on $S$. The complete graph on $n$ vertices is denoted by $K_n$. Our notation is intended to be consistent with [10].

A subset of $V(G)$ is *independent* if no two vertices in the subset are adjacent. A *maximum independent set* is an independent set of largest cardinality. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set. Finding a maximum independent set (MIS) in a graph is a well-known and widely-studied NP-hard problem [7].

A *matching* in $G$ is a set of pairwise non-incident edges, and the *matching number* $\mu(G)$ is the cardinality of a maximum matching. A graph $G$ is called *König-Egervary* if $\alpha(G) + \mu(G) = n(G)$. Every bipartite graph is a *König-Egervary graph*, but not conversely.

The *Cartesian product* of graphs $G$ and $H$ is the graph $G \Box H$, whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are the pairs $(g, h)(g', h')$ for which one of the following holds:

1. $g = g'$ and $hh' \in E(H)$ or,
2. $gg' \in E(G)$ and $h = h'$.

The product $G \Box H$ is connected if and only if both $G$ and $H$ connected [10, Proposition 1.34], and it is bipartite if and only if both $G$ and $H$ are bipartite. Moreover, if $C_G$ and $C_H$ are induced bipartite subgraphs of $G$ and $H$ respectively, then $C_G \Box C_H$ is an induced bipartite subgraph of $G \Box H$ with $n(C_G)n(C_H)$ vertices. By [11, p. 102], the radius of $G \Box H$ is $r(G \Box H) = r(G) + r(H)$.

We will have occasion to use a class of graphs called $r$-ciliates. For positive integers $q$ and $r \geq q$, the *$r$-ciliate* $C_{2q,r-q}$ is the graph obtained by appending to each vertex of the even cycle $C_{2q}$ a path on $r - q$ vertices. Figure 1 shows $r$-ciliates for various values of $q$ and $r$. Notice that when $q = 1$ the cycle is degenerate and identical to $K_2$, so the $r$-ciliate is a path on $2r$ vertices. In the case where $r = q$, the $r$-ciliate is a cycle on $2r$ vertices. It is easy to see that each $r$-ciliate has radius $r$, is bipartite, has independence number $\alpha(C_{2q,r-q}) = \frac{1}{2}n(C_{2q,r-q})$, and has a perfect matching.

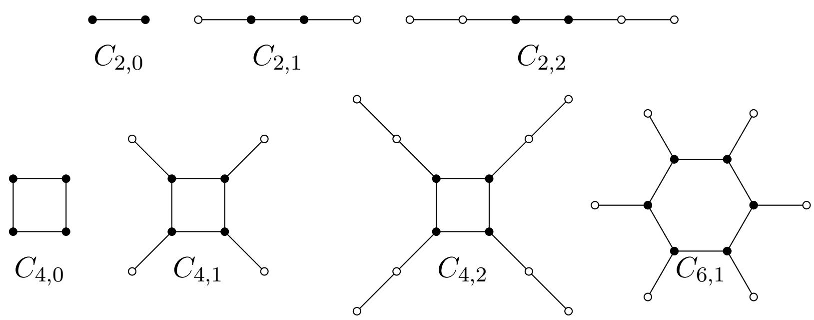


Figure 1. Examples of $r$-ciliates.

A connected graph $G$ is *radius-critical* if $r(G - v) < r(G)$ for each $v \in V(G)$ that is not a cut vertex. Fajtlowicz classified radius-critical graphs as follows.

**Theorem 1.1** (Fajtlowicz, [5]). *A connected graph of radius $r$ is radius-critical if and only if it is an $r$-ciliate.*

This implies every connected graph of radius $r$ contains an induced $r$-ciliate, for we can obtain this subgraph by successively deleting vertices that do not decrease the radius, until this is no longer possible.

The *bipartite number* $\alpha_2(G)$ of a graph $G$ is the order of the largest induced bipartite subgraph of $G$. As an $r$-ciliate of $G$ has at least $2r(G)$ vertices, it follows that $\alpha_2(G) \geq 2r(G)$. We will need the following result concerning bipartite numbers of graphs.

**Theorem 1.2** ([11, Proposition 7.3]). *If $G$ and $H$ are graphs and $G$ is bipartite, then $\alpha(G \Box H) \geq \frac{1}{2}n(G)\alpha_2(H)$, and equality holds if $G$ has a perfect matching.*

## 2. Radius, Independence and the Cartesian Product

Siemion Fajtlowicz's computer program Graffiti [6] conjectured that $\alpha(G) \geq r(G)$ for any connected graph $G$. This conjecture follows immediately from the Induced Path Theorem, which was proved by Erdös, Saks and Sós, [4, Theorem 2.1], using an approach credited to Fan Chung. Fajtlowicz [5] mentions four different proofs of this conjecture as of 1988.

Graphs for which the equality $\alpha(G) = r(G)$ is attained are worthy of special attention. In this section we characterize the situations under which $\alpha(G \Box H) = r(G \Box H)$. The following lemma will be needed for our main result.

**Lemma 2.1.** *If $G$ and $H$ are connected graphs, then $\alpha(G \Box H) \geq 2r(G)r(H)$.*

***Proof.*** Let $C_G$ be an induced $r(G)$-ciliate of $G$ and $C_H$ be an induced $r(H)$-ciliate of $H$. Then $C_G$ and $C_H$ are bipartite, they each have perfect matchings and their independence numbers are $\alpha(C_G) = \frac{1}{2}n(C_G)$ and $\alpha(C_H) = \frac{1}{2}n(C_H)$ respectively. Moreover, as $C_G \Box C_H$ is bipartite, we have $\alpha(C_G \Box C_H) \geq \frac{1}{2}n(C_G \Box C_H) = \frac{1}{2}n(C_G)n(C_H)$. Thus

$$\alpha(G \Box H) \geq \alpha(C_G \Box C_H) \geq \frac{1}{2}n(C_G)n(C_H) \geq \frac{1}{2}2r(G)2r(H) = 2r(G)r(H).$$

The last inequality follows from the fact that $r$-ciliates have at least $2r$ vertices. ■

Lemma 2.1 is interesting by itself, as it gives a new efficiently computable lower bound for $\alpha(G\Box H)$ (this is due to the fact that the radius of a connected graph can be efficiently computed). For now, we use it to prove the following result.

**Theorem 2.2.** *If $G$ and $H$ are connected graphs, then $\alpha(G\Box H) = r(G\Box H)$ if and only if one of the following conditions holds.*

1. *$G = K_1$ and $\alpha(H) = r(H)$ (or symmetrically, $H = K_1$ and $\alpha(G) = r(G)$).*
2. *$G = K_2$ and $H = K_n$ (or symmetrically, $H = K_2$ and $G = K_n$).*

***Proof.*** Suppose $\alpha(G\Box H) = r(G\Box H)$. If $G$ or $H$ (say $G$) is $K_1$, then $G\Box H \cong H$ and $\alpha(H) = \alpha(G\Box H) = r(G\Box H) = r(G) + r(H) = r(H)$, so Condition (1) holds.

Now assume that neither $G$ nor $H$ is $K_1$. Then both $G$ and $H$ have radius greater than zero. Lemma 2.1 now yields

$$r(G) + r(H) = r(G\Box H) = \alpha(G\Box H) \geq 2r(G)r(H).$$

Since $r(G)$ and $r(H)$ are integers greater than zero, and $r(G) + r(H) \geq 2r(G)r(H)$, it follows that $r(G) = r(H) = 1$. So $r(G\Box H) = \alpha(G\Box H) = 2$. Let $V(G) = \{v_1, v_2, \ldots, v_g\}$ and $V(H) = \{w_1, w_2, \ldots, w_h\}$. If $g \geq h$ then the pairs $(v_1, w_1), (v_2, w_2), \ldots, (v_h, w_h)$ are well-defined, and independent (since none of their coordinates are the same pair-wise, they cannot be adjacent, by the definition of the Cartesian graph product). This shows that $\alpha(G\Box H) \geq \min\{n(G), n(H)\}$. (Much stronger results are known. See [13].) Thus $2 \geq \min\{n(G), n(H)\}$, so either $n(G) = 2$ or $n(H) = 2$. We may assume the former and, since $G$ is connected, we have $G = K_2$.

Now let $B_H$ be a maximum induced bipartite subgraph of $H$, so $\alpha_2(H) = n(B_H)$. Since $K_2$ has a perfect matching, Theorem 1.2 produces $\alpha(K_2\Box H) = \frac{1}{2}n(K_2)\alpha_2(H) = \alpha_2(H)$, so $\alpha_2(H) = 2$. This means $n(H) \geq 2$ and any set of three vertices of $H$ induces a triangle, so $H = K_n$ for some $n \geq 2$.

Conversely, suppose either (1) $G = K_1$ and $\alpha(H) = r(H)$, or (2) $G = K_2$ and $H = K_n$ for $n \geq 2$. In the first case

$$\alpha(G\Box H) = \alpha(K_1\Box H) = \alpha(H) = r(H)$$

$$= r(K_1) + r(H) = r(K_1\Box H) = r(G\Box H).$$

For the second case, an application of Theorem 1.2 gives

$$\begin{aligned}\alpha(G\Box H) = \alpha(K_2\Box K_n) = \frac{1}{2}n(K_2)\alpha_2(K_n) = 2 = r(K_2) + r(K_n)\\ = r(K_2\Box K_n) = r(G\Box H).\end{aligned}$$ ■

Figure 2 shows an instance of Theorem 2.2. Note $\alpha(K_2\Box K_3) = r(K_2\Box K_3) = 2$.

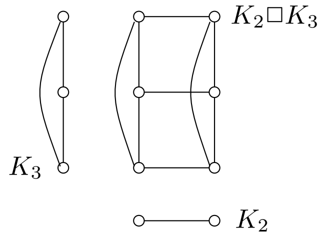


Figure 2. An illustration of Theorem 2.2.

Lemma 2.1 states that $\alpha(G\Box H) \geq 2r(G)r(H)$ for connected graphs $G$ and $H$. Except in the case where $G$ or $H$ is the trivial graph, this lower bound is at least as good as $r(G\Box H)$ (as can easily be checked). This raises the natural question of characterizing those graphs for which $\alpha(G\Box H) = 2r(G)r(H)$, and we now turn our attention to that task.

**Theorem 2.3.** *Let $G$ and $H$ be connected graphs. Then $\alpha(G\Box H) = 2r(G)r(H)$ if and only if at least one of the following holds:*

1. *$G$ is a path or cycle on $2r(G)$ vertices, and $\alpha_2(H) = 2r(H)$, or*
2. *$H$ is a path or cycle on $2r(H)$ vertices, and $\alpha_2(G) = 2r(G)$.*

***Proof.*** Suppose $G$ is a path or cycle on $2r(G)$ vertices and $\alpha_2(H) = 2r(H)$. Then $G$ is bipartite with a perfect matching. Theorem 1.2 yields $\alpha(G\Box H) = \frac{1}{2}n(G)\alpha_2(H) = 2r(G)r(H)$.

Conversely suppose that $\alpha(G\Box H) = 2r(G)r(H)$. Let $G_B$ and $H_B$ be maximum induced bipartite subgraphs of $G$ and $H$, respectively, so $\alpha_2(G) = n(G_B)$ and $\alpha_2(H) = n(H_B)$. Observe $\alpha_2(G) \geq 2r(G)$, because any ciliate of $G$ is an induced bipartite subgraph and has at least $2r(G)$ vertices. Using this and Theorem 1.2, we get

$$\alpha(G\Box H) \geq \alpha(G_B\Box H) \geq \frac{1}{2}n(G_B)\alpha_2(H) \geq \frac{1}{2}\alpha_2(G)\alpha_2(H) \geq 2r(G)r(H).$$

The assumption implies that all of these terms are equal, so $\alpha_2(G)\alpha_2(H) = 4r(G)r(H)$. Since $\alpha_2(G) \geq 2r(G)$ and $\alpha_2(H) \geq 2r(H)$, we have $\alpha_2(G) = 2r(G)$ and $\alpha_2(H) = 2r(H)$. If $n(G) = \alpha_2(G) = 2r(G)$ then $G$ is a $2r$-path or cycle, and we have Condition 1. Similarly, if If $n(H) = \alpha_2(H) = 2r(H)$, then $H$ is a $2r$-path or cycle and we have Condition 2.

We complete the proof by showing that it is not possible to have both $n(G) > 2r(G)$ and $n(H) > 2r(H)$. Suppose to the contrary that both of these hold. Put $G' = G\langle V(G) - V(G_B)\rangle$ and $H' = H\langle V(H) - V(H_B)\rangle$, so both $G'$ and $H'$ are non-empty. Note that, by definition, no vertex of $G_B \square H_B$ is adjacent to any vertex of $G' \square H'$. Then

$$\begin{aligned}\alpha(G\square H) &\geq \alpha(G_B\square H_B) + \alpha(G'\square H') \geq \frac{1}{2}n(G_B)n(H_B) + \alpha(G'\square H') \\ &> \frac{1}{2}n(G_B)n(H_B) = \frac{1}{2}\alpha_2(G)\alpha_2(H) = 2r(G)r(H)\end{aligned}$$

contradicting the assumption that the first and last terms are equal. ■

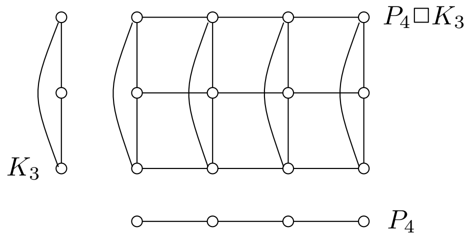


Figure 3. An illustration of Theorem 2.3. Here $\alpha(K_3 \square P_4) = 2r(K_3)r(P_4) = 4$, and $P_4$ is a $2r$-path, and $\alpha_2(K_3) = 2r(K_3) = 2$.

## 3. Independence Irreducibility and the Cartesian Product

For any $S \subseteq V(G)$, let $N_G(S)$ be the the set of neighbors of $S$. An independent set of vertices $I$ is a *critical independent set* if $|I| - |N(I)|$ is maximized. A *maximum critical independent set* is a critical independent set of maximum cardinality. The *critical independence number* of a graph $G$, denoted $\alpha'(G)$, is the cardinality of a maximum critical independent set. If $I$ is a maximum critical independent set, so $\alpha'(G) = |I|$, then clearly $\alpha'(G) \leq \alpha(G)$. The critical independence number can be computed in polynomial-time [14]. This immediately yields a new polynomial-time computable lower bound for the Cartesian product: $\alpha(G\Box H) \geq \alpha(G)\alpha(H) \geq \alpha'(G)\alpha'(H)$. Research on critical independent sets was initiated by Zhang [19] and most recently advanced by Butenko and Trukhanov [2].

A graph is *independence irreducible* if every non-empty independent set $I$ has more than $|I|$ neighbors. (Fullerene graphs, for instance, are independence irreducible [14].) For these graphs, $\alpha'(G) = 0$. A graph is *independence reducible* if $\alpha'(G) > 0$. A graph is *totally independence reducible* if $\alpha'(G) = \alpha(G)$. ($K_2$ is an example.) This class of graphs was shown in [15, 16] to be equivalent to the class of König-Egervary graphs. Deciding whether a graph is totally independence reducible can be done in polynomial-time [15, 16]. The following structural theorem states that any graph can be decomposed into two induced subgraphs, one Koňig-Egervary and the other independence irreducible.

**Theorem 3.1** (Larson, [15, 16])**.** *For any graph $G$, there is a unique set $X \subseteq V(G)$ such that*

1. $\alpha(G) = \alpha(G\langle X\rangle) + \alpha(G\langle V(G) - X\rangle)$,
2. $G\langle X\rangle$ *is Koňig-Egervary* (*or totally independence reducible*),
3. $G\langle V(G) - X\rangle$ *is independence irreducible, and*
4. *for every maximum critical independent set $J_c$ of $G$, $X = J_c \cup N(J_c)$.*

It seems reasonable to ask if this result in any way respects the factors of a Cartesian product. We conclude with some results and conjectures in this direction.

**Theorem 3.2.** *If $G$ or $H$ is independence irreducible, then $G\Box H$ is independence irreducible.*

***Proof.*** Suppose that $G$ is independence irreducible. Let $I$ be an independent set in $G\Box H$. Let $p_H(I)$ be the projection of $I$ onto $H$, that is $p_H(I) = \{y \in H | (x, y) \in I\}$. For every $y \in H$ let $G_y = \{(x, y)|x \in G\}$. For every $y \in p_H(I)$, let $I_y = G_y \cap I$. Thus $I_y$ is a subset of $I$, and is therefore independent, and, moreover, $p_G(I_y)$, is independent in $G$. Furthermore, the sets $I_y$ form a partition of $I$. Note that, since $G$ is independence irreducible, for every $y \in p_H(I)$ we have $|N_G(p_G(I_y))| > |p_G(I_y)|$.

Now, for every $y \in p_H(I)$, let $N_y = N_G(p_G(I_y)) \times \{y\} \subseteq G_y$. The sets $N_y$ are, by construction, disjoint subsets of $N_{G\Box H}(I)$, so $\sum_{y\in p_H(I)} |N_y| \leq$

$|N_{G\square H}(I)|$. Thus

$$|I| = \sum_{y \in p_H(I)} |I_y| = \sum_{y \in p_H(I)} |p_G(I_y)| < \sum_{y \in p_H(I)} |N_G(p_G(I_y))|$$
$$= \sum_{y \in p_H(I)} |N_y| \leq |N_{G\square H}(I)|,$$

which was to be proved. ■

For an illustration of this theorem, note that $K_3$ is independence irreducible, and it is easy to check that the graph $K_2 \square K_3$ in Figure 2 is independence irreducible.

The converse of Theorem 3.2 is false, and Figure 4 shows a counterexample. Both factors are totally independence reducible but the product is independence irreducible (as can be checked by hand).

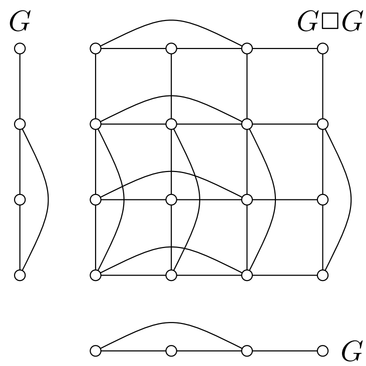


Figure 4. An example of a König-Egervary graph $G$ for which the product $G \square G$ is not König-Egervary. Note $\alpha(G) + \mu(G) = 2 + 2 = n(G)$, and $\alpha(G \square G) + \mu(G \square G) = 6 + 8 \neq n(G \square G) = 16$.

We close with several questions. First, what conditions on a Cartesian product guarantee that one factor must be independence irreducible?

The Cartesian product of bipartite graphs is bipartite, and all bipartite graphs are König-Egervary. (In this sense König-Egervary graphs are generalizations of bipartite graphs.) It is reasonable then to conjecture that the Cartesian product of König-Egervary graphs is König-Egervary. But this is not the case, as Figure 4 demonstrates. We ask what conditions on $G$ and $H$ imply $G \square H$ is König-Egervary.

Finally the statement of Theorem 2.3 involves graphs for which $\alpha_2(G) = 2r(G)$. It would be interesting to characterize this class of graphs.

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