A NOTE ON CRITICAL INDEPENDENCE REDUCTIONS

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Abstract. A criterion is specified for identifying graphs with nonempty critical independent sets. A polynomial-time algorithm is given for finding them and, thus, reducing the problem of finding a maximum independent set (MIS) in such a graph to finding a MIS in a proper subgraph. This algorithm can be extended to identify maximum cardinality critical independent sets, answering a question of Butenko and Trukhanov.

1. Introduction

An independent set of vertices in a graph is a set of vertices no two of which are adjacent. A maximum independent set is a independent set of largest cardinality. Finding a maximum independent set (MIS) in a graph is a well-known widely-studied NP-hard problem [7]. In this note we describe a polynomial-time algorithm for reducing this problem to the MIS problem on a subgraph.

The following notation is used throughout: the vertex set of a graph G is $V(G)$, the order $n(G) = |V(G)|$, the edge set is $E(G)$, the set of neighbors of a set $S \subseteq V(G)$ is $N(S)$, the graph induced on a set $S \subseteq V(G)$ is $G[S]$, and the independence number, the cardinality of a MIS, is $\alpha(G)$. All graphs are assumed to be finite and simple.

An independent set of vertices I is a *critical independent set* if $|I|-|N(I)|$ is maximized. Butenko and Trukhanov proved that any critical independent set is contained in a maximum independent set [5]. This can lead to a speed-up of the problem of finding a maximum independent set (MIS) and the independence number of a graph: if I is a critical independent set of a graph G, then the problem of finding a MIS can be reduced to finding one for $G \setminus (I \cup N(I))$. In fact, Butenko and Trukhanov demonstrate that the speed-up from this reduction can be dramatic.

The algorithm Butenko and Trukhanov use for finding a critical independent set, first suggested by Ageev [1], does not always result in a non-empty critical independent set in cases where there is, in fact, such a set, and thus does not always result in a reduction of the problem of finding a MIS to a smaller graph. In this note, a criterion will be specified for when a non-empty critical independent set exists as well as an algorithm for finding one in polynomial-time.

Butenko and Trukhanov ask "how to find the largest critical independent set in a graph?" This question can now be answered. The specified algorithm can be extended to yield a maximum-cardinality critical independent set.

2. Critical Independent Sets

Definition 2.1. $C \subseteq V(G)$ is a *critical set* of a graph G if $|C| - |N(C)| \geq |U| - |N(U)|$ for every $U \subseteq V(G)$.

Definition 2.2. $I \subseteq V(G)$ is a *critical independent set* of a graph G if I is an independent set of vertices and $|I| - |N(I)| \geq |U| - |N(U)|$ for every independent set $U \subseteq V(G)$.

A graph may contain critical independent sets of different cardinalities. A graph consisting of a single edge (K_2, the complete graph on two vertices) has critical independent sets of cardinalities 0 and 1. A graph may not contain a non-empty critical independent set. For instance, the empty set is the unique critical independent set of K_3. In fact, for any graph with a perfect matching (which is, in a well-defined sense, almost every graph with an even number of vertices [3, p. 178]), the empty set is a critical independent set. Finding a critical independent set using Ageev's algorithm may yield no reduction in these cases. There are, though, graphs with perfect matchings which have non-empty critical independent sets: K_2 is an example.

We now define the *bi-double graph* of a given graph. This graph is utilized in a proof of Zhang [8], and referred to in the papers of Ageev and Butenko and Trukhanov, and is a more generally useful proof technique (see, for instance, [2]).

Definition 2.3. For a graph G, the *bi-double* graph $B(G)$ has vertex set $V \cup V'$, where V' is a copy of V. If $V = \{v_1, v_2, \ldots, v_n\}$, let $V' = \{v'_1, v'_2, \ldots, v'_n\}$. Then, $(x, y') \in E(B(G))$ if, and only if, $(x, y) \in E(G)$.

Theorem 2.4. *(Ageev) If C is a critical set then the isolated points in $G[C]$, the graph induced on C, is a critical independent set.*

Theorem 2.5. *(Ageev) For a graph G, if I is a maximum independent set in the bi-double graph $B(G)$, then $U = V(G) \cap I$ is a critical set for G.*

The preceding two theorems imply that the following algorithm of Ageev results in a critical independent set I in a graph G:

(1) Construct the bi-double graph $B(G)$ of G.
(2) Find a maximum independent set J in $B(G)$.
(3) Let $J' = V \cap J$.
(4) Let I be the set of isolated points in $G[J']$.

Since a maximum independent set in a bipartite graph can be found in polynomial-time, Ageev's algorithm yields a critical independent set in polynomial time. (Zhang was the first to prove the existence of such an algorithm).

Butenko and Trukhanov showed that identifying a non-empty critical independent set gives a polynomial-time reduction of the problem of finding a maximum independent set to a proper subgraph. The following theorem identifies a fact about the structure of critical independent sets, and leads to a new proof (below) of Butenko and Trukhanov's theorem.

Definition 2.6. For disjoint subsets X and Y of the vertices of a graph G, there is a *matching from X to Y* if there is a set of disjoint edges having one endpoint in X and the other in Y and that covers all of the vertices in X.

Theorem 2.7. *If I is a critical independent set, then there is a matching from $N(I)$ to I.*

Proof. Note that I and $N(I)$ are disjoint. Let B be the subgraph of G whose vertices are $I \cup N(I)$, with $(x, y) \in E(B)$ if, and only if, $x \in I$, $y \in N(I)$, and $(x, y) \in E(G)$. Clearly, it is enough to prove the claim for this subgraph. Let $J \subseteq N(I)$. Suppose, $|J| > |N(J)|$. Let $X = I - N(J)$. Then $N(X) \subseteq N(I) - J$ (so $|N(I)| \geq |N(X)| + |J|$), and

$$|X| - |N(X)| > |X| - |N(X)| - (|J| - |N(J)|) =$$

$$(|X| + |N(J)|) - (|N(X)| + |J|) \geq |I| - |N(I)|.$$

This contradicts the fact that I is a critical independent set. So, it must be that $|N(J)| \geq |J|$. Since this is true for every subset $J \subset N(I)$, Hall's Theorem (see, for instance, [4]) implies the claim. □

Theorem 2.8. *(Butenko & Trukhanov) If I_c is a critical independent set of a graph G, then there exists a maximum independent set I of G, such that $I_c \subseteq I$.*

Proof. Let I_c be a critical independent set of G. Let J be a maximum independent set of G, and let $J_N = J \cap N(I_c)$. Let $J'_N \subseteq I_c$ be the vertices matched with vertices of J_N by the matching from $N(I_c)$ to I_c given by Theorem 2.7. Let $J' = (J \setminus J_N) \cup J'_N$. Clearly, $J \cap J'_N$ is empty, $|J_N| = |J'_N|$, J' is an independent set in G, and $|J| = |J'|$. So J' is a maximum independent set. Since the vertices in $I_c \setminus J'_N$ are adjacent only to elements of $N(I_c)$, and $J' \cap N(I_c)$ is empty, $I_c \setminus J'_N \subseteq J'$. Since, by definition, $J'_N \subseteq J'$, it follows that $I_c \subseteq J'$. □

3. A criterion for independence reductions

If there is *any* non-empty independent set I such that $|I| \geq |N(I)|$, then there is a non-empty critical independent set. If there is a pendant for instance, there is a non-empty critical independent set. In this sense, and in the sense that identification and removal of these sets reduces the problem of finding maximum independent sets, a critical independent set can be viewed as a generalization of a pendant.

Ageev's algorithm yields a critical independent set—but this set may be empty in the case where non-empty independent sets exist. The following results imply an algorithm which yields a non-empty critical independent set when one exists.

Lemma 3.1. *If I_c is a critical independent set of the graph G then $I = I_c \cup (V' - N(I_c))$ is a maximum independent set of the bi-double graph $B(G)$.*

Proof. Clearly I is independent. Suppose there is a maximum independent set $J \subseteq V(B(G))$ such that $|J| > |I|$. Let $J_V = J \cap V$ and $J_{V'} = J \cap V'$. So, $|J_V| + |J_{V'}| = |J| > |I| = |I_c| + |V' \setminus N(I_c)|$. Since $J_{V'} = V' \setminus N(J_V)$, $|V' \setminus N(J_V)| = |V| - |N(J_V)|$, and $|V' \setminus N(I_c)| = |V| - |N(I_c)|$, it follows that $|J_V| + |V| - |N(J_V)| > |I_c| + |V| - |N(I_c)|$. By Theorem 2.5, J_V is a critical set in G and, thus, that I_c is not a critical set (nor a critical independent set).

Thus, since critical independent sets are critical sets, I_c is not a critical set. Thus, I is a maximum independent set of $B(G)$. □

Theorem 3.2. *A graph G contains a non-empty critical independent set if, and only if, there is a maximum independent set of the bi-double graph $B(G)$ containing both v and v', for some vertex $v \in V(G)$, and its copy $v' \in V'$.*

Proof. If I_c is a non-empty critical independent set of G then, by Lemma 3.1, $I = I_c \cup (V' - N(I_c))$ is a maximum independent set of $B(G)$. For any vertex $v \in I_c$, v is not adjacent to v' in $B(G)$. Thus, $v' \in V' - N(I_c)$. Thus v and v' are in I.

Suppose I is a maximum independent set of $B(G)$ containing both v and v'. Then $v \in J = I \cap V(G)$. By Theorem 2.5, J is a critical set in G, and by Theorem 2.4, the isolated points in $G[J]$ are a critical independent set in G. Suppose v is not an isolated point in $G[J]$. Then there is a vertex $w \in J$ such that v is adjacent to w in G. This implies that w is adjacent to

v' in $B(G)$. So I contains v' and w and I is independent, contradicting the assumption that v is not isolated in $G[J]$. Thus, the set of isolated points of $G[J]$ is non-empty. □

Corollary 3.3. *A graph G contains a non-empty critical independent set if, and only if, there is a vertex $v \in V(G)$ such that $\alpha(B(G)) = \alpha(B(G) - \{v, v'\} - N(\{v, v'\}) + 2$.*

Proof. This follows immediately from Theorem 3.2. □

Corollary 3.3 suggests a polynomial-time algorithm for finding a non-empty critical independent set in a graph if one exists:

(1) Construct graph $B(G)$.
(2) Set BOOL=false.
(3) For $i = 1, \ldots, n = |V(G)|$, set BOOL=true if $\alpha(B(G)) = \alpha(B(G) - \{v_i, v_i'\} - N(\{v_i, v_i'\}) + 2$. If BOOL=true, break.
(4) If BOOL=true,
   (a) Find a maximum independent set J in $B(G) - \{v_i, v_i'\} - N(\{v_i, v_i'\})$.
   (b) Let $J' = J \cap V$.
   (c) Let I be the set of isolated points in $G[J']$ together with v.

If BOOL=true, I is a non-empty critical independent set. If BOOL=false, then no non-empty critical independent set exists in G.

4. Maximum Critical Independent Sets

Definition 4.1. A critical independent set is *maximal* if there is no critical independent set which properly contains it. It is *maximum* if there is no critical independent set with larger cardinality.

Butenko and Trukhanov raised the question of how to identify maximum critical independent sets. These sets will result in a maximum reduction in the problem of finding a MIS. The following theorem justifies an algorithm that yield these sets.

Theorem 4.2. *Any critical independent set is contained in a maximum critical independent set.*

Proof. Suppose I is a critical independent set and J is a maximum critical independent set. Let $I' = I \cup J'$, where $J' = J \setminus (I \cup N(I))$. It is enough to show that I' is a maximum critical independent set. Clearly I' is independent.

We will first show that I' is a critical (independent) set; in particular, that $|I'|-|N(I')| \geq |I|-|N(I)|$. Since I and J' are disjoint, $|I'| = |I|+|J'|$. $N(I') \subseteq N(I) \cup [N(J) \setminus (I \cup N(I))]$ and $|N(I')| \leq |N(I)|+|N(J) \setminus (I \cup N(I))|$ So,

$$|I'| - |N(I')| \geq |I| - |N(I)| + |J'| - |N(J) \setminus (I \cup N(I))|.$$

It is enough then to show, $|J'| \geq |N(J) \setminus (I \cup N(I))|$.

Now, $J = J' \cup (I \cap J) \cup (N(I) \cap J)$. By definition, J', $I \cap J$, and $N(I) \cap J$ are mutually disjoint. So,

$$|J| = |J'| + |I \cap J| + |N(I) \cap J|.$$

Also, $N(J) = (I \cap N(J)) \cup (N(I) \cap N(J)) \cup (N(J) \setminus (I \cup N(I))$. Clearly, $I \cap N(J)$, $N(I) \cap N(J)$ and $N(J) \setminus (I \cup N(I))$ are disjoint. So,

$$|N(J)| = |I \cap N(J)| + |N(I) \cap N(J)| + |N(J) \setminus (I \cup N(I))|.$$

Then,

$$|J| - |N(J)| = \tag{4.1}$$

$$(|N(I) \cap J| - |N(J) \cap I|) + |I \cap J| + |J'| - (|N(I) \cap N(J)| + |N(J) \setminus (I \cup N(I))|).$$

Now, Theorem 2.7 guarantees that there is a matching from $N(J)$ to J and from $N(I)$ to I. Since the vertices in $I \cap N(J) \subseteq N(J)$ must be matched to vertices in $N(I) \cap J$, and the vertices in $N(I) \cap J \subseteq N(I)$ must be matched to vertices in $I \cap N(J)$, it follows that $|I \cap N(J)| = |J \cap N(I)|$ and $|N(I) \cap J| - |N(J) \cap I|$, the first term in Equation 4.1, is 0.

Assume that $|N(J) \setminus (I \cup N(I))| > |J'|$. Note that $N(I \cap J) \subseteq N(I) \cap N(J)$ and $|N(I \cap J)| \leq |N(I) \cap N(J)|$. Then Equation 4.1 gives,

$$\begin{aligned} |J| - |N(J)| &= |I \cap J| + |J'| - (|N(I) \cap N(J)| + |N(J) \setminus (I \cup N(I))|) \\ &< |I \cap J| - |N(I) \cap N(J)| \leq |I \cap J| - |N(I \cap J)|, \end{aligned}$$

contradicting the fact that J is a critical (independent) set. Thus, $|N(J) \setminus (I \cup N(I)| \leq |J'|$ and I' is a critical independent set.

Lastly, we show that I' is maximum; in particular that $|I'| = |J|$. It was noted above that $J = (J \cap I) \cup (J \cap N(I)) \cup J'$, $|J| = |J \cap I| + |J \cap N(I)| + |J'|$, and $|I'| = |I| + |J'|$. Since I is a critical independent set, by Theorem 2.7 there is a matching from $N(I)$ to I. Let $J'_N \subseteq I$ be the vertices matched to $(J \cap N(I)) \subseteq N(I)$ under this matching. So $|J'_N| = |J \cap N(I)|$. Clearly J'_N and $J \cap I$ are disjoint. $|I'| = |I| + |J'| = |J'_N| + |I \setminus J'_N| + |J'| \geq |J \cap N(I)| + |J \cap I| + |J'| = |J|$. Note that $(J \cap I) \subseteq (I \setminus J'_N)$. So $|I'| \geq |J|$ and, since J is a maximum critical independent set, $|I'| = |J|$. Thus I' is a maximum critical independent set.

□

Corollary 4.3. *If a vertex v of a graph G is contained in some critical independent set, then there is a maximum critical independent set which contains v.*

Corollary 4.4. *A maximal critical independent set is a maximum critical independent set.*

The idea of the following algorithm is to find a maximal critical independent set I by choosing a vertex, testing if it is contained in a critical independent set and, if it is, adding it to I and removing it and its neighbors. In either case the process is repeated on the graph induced on the remaining vertices.

The Maximal Critical Independent Set (MCIS) Algorithm

(1) Construct bi-double graph $B(G)$ of G.
(2) $i := 1$, $I := \emptyset$;
(3) If $i > |V(G)|$, return I.
(4) If $v_i \notin V(B(G))$, $i := i + 1$, and return to Step 3.
(5) If $\alpha(B(G)) = \alpha(B(G) - \{v_i, v_i'\} - N(\{v_i, v_i'\})$, BOOL:=true. Else, BOOL:=false.
(6) If BOOL=true, $I := I \cup \{v_i\}$, $V(B(G)) := V(B(G)) \setminus (\{v_i, v_i'\} \cup N(\{v_i, v_i'\})$, $i := i + 1$. Return to Step 3.
(7) If BOOL=false, $V(B(G)) := V(B(G)) \setminus \{v_i, v_i'\}$, $i := i + 1$. Return to Step 3.

Theorem 4.5. *The MCIS Algorithm yields a maximum critical independent set.*

Proof. By Corollary 4.4 it is enough to show that this algorithm produces a maximal critical independent set. For graphs with a single vertex, the MCIS algorithm returns a one-element set. This set is clearly a maximal critical independent set.

Suppose the MCIS algorithm produces a maximal critical independent set for graphs with n or fewer vertices. Suppose G has $n+1$ vertices. If the only critical independent set of G is the empty set then, by Corollary 3.3, the test in Step 6 will be negative and set I will remain empty after each loop. I is a maximal critical independent set.

Suppose G contains a non-empty critical independent set. Let i be the first index so that v_i belongs to a critical independent set. Corollary 3.3 guarantees that the test in Step 6 will be negative. The MCIS algorithm then sets $I := \{v_i\}$ and continues on the graph G' induced on $V(G)-\{v_i\}-N(\{v_i\})$. This graph has n or fewer vertices. By assumption, the MCIS algorithm (then) yields a maximal critical independent set J for G'. So $I = J \cup \{v_i\}$ is a maximal critical independent set for G. □

Theorem 4.6. *If I is a maximum critical independent set of G, then the only critical independent set of $G - I - N(I)$ is the empty set.*

This means that any further repetition of the MCIS algorithm will not yield any further reduction.

5. Weighted Critical Independent Sets

The results of Ageev, Butenko and Trukhanov all have analogues for weighted graphs. An anonymous referee asked "whether these results can be extended to the weighted version of the MIS problem as well." The answer is *yes and maybe*: a criterion can be specified for the existence of a critical weighted independent set which exactly parallels the criterion for existence in the non-weighted case, but it is an open question whether any critical weighted independent set is contained in a maximum weight independent set.

Definition 5.1. A *weighted graph* is a graph where a nonnegative number w_i (called a *weight*) is associated to each vertex v_i. The weight of a set $S \subseteq V$ is $w(S) = \sum_{v_i \in S} w_i$. S is a *critical weighted set* if

$$w(S) - w(N(S)) \geq w(T) - w(N(T))$$

for any set $T \subseteq V$. A set $I \subseteq V$ is a *critical weighted independent set* if I is independent and a critical weighted set. I is a *maximum weight independent set* if I is independent and $w(I) \geq w(J)$ for any other independent set J. A critical weighted independent set is *maximum* if there is no critical weighted independent set with larger weight.

The weighted extensions of Ageev's theorems [1] are reproduced here as they are required for the proof of Theorem 5.5.

Theorem 5.2. *(Ageev) If C is a critical weighted set then the isolated points in $G[C]$, the graph induced on C, is a critical weighted independent set.*

Theorem 5.3. *(Ageev) For a graph G, if I is a maximum weighted independent set in the bi-double graph $B(G)$, then $U = V(G) \cap I$ is a critical weighted set for G.*

The criterion for determining if a graph has a non-empty critical independent set can be extended to the weighted case: there is a polynomial-time criterion for determining if a weighted graph has a non-empty critical weighted independent set.

Lemma 5.4. *If I_c is a critical weighted independent set of the graph G then $I = I_c \cup (V' - N(I_c))$ is a maximum weight independent set of the bi-double graph $B(G)$.*

Proof. Clearly I is independent. Suppose there is a maximum weight independent set $J \subseteq V(B(G))$ such that $w(J) > w(I)$. Let $J_V = J \cap V$ and $J_{V'} = J \cap V'$. So, $w(J_V) + w(J_{V'}) = w(J) > w(I) = w(I_c) + w(V' \setminus N(I_c))$. Since $J_{V'} = V' \setminus N(J_V)$, $w(V' \setminus N(J_V)) = w(V) - w(N(J_V))$, and $w(V' \setminus N(I_c)) = w(V) - w(N(I_c))$, it follows that $w(J_V) + w(V) - w(N(J_V)) > w(I_c) + w(V) - w(N(I_c))$. By Theorem 5.3, J_V is a critical weighted set in

G and, thus, that I_c is not a critical weighted set (nor a critical weighted independent set).

Thus, since critical weighted independent sets are critical weighted sets, I_c is not a critical weighted set. Thus, I is a maximum weighted independent set of $B(G)$. □

Theorem 5.5. *A graph G contains a non-empty critical weighted independent set if, and only if, there is a maximum weight independent set of the bi-double graph $B(G)$ containing both v and v', for some vertex $v \in V(G)$, and its copy $v' \in V'$.*

Proof. If I_c is a non-empty critical weighted independent set of G then, by Lemma 5.4, $I = I_c \cup (V' - N(I_c))$ is a maximum weighted independent set of $B(G)$. For any vertex $v \in I_c$, v is not adjacent to v' in $B(G)$. Thus, $v' \in V' - N(I_c)$. Thus v and v' are in I.

Suppose I is a maximum weight independent set of $B(G)$ containing both v and v'. Then $v \in J = I \cap V(G)$. By Theorem 5.3, J is a critical weighted set in G, and by Theorem 5.2, the isolated points in $G[J]$ are a critical weighted independent set in G. Suppose v is not an isolated point in $G[J]$. Then there is a vertex $w \in J$ such that v is adjacent to w in G. This implies that w is adjacent to v' in $B(G)$. So I contains v' and w and I is independent, contradicting the assumption that v is not isolated in $G[J]$. Thus, the set of isolated points of $G[J]$ is non-empty. □

If I is a maximum weight independent set of a graph G, let the weighted independence number of G be $\alpha_w(G) = w(I)$. Since every graph has a maximum weight independent set, $\alpha_w(G)$ is well-defined.

Corollary 5.6. *A graph G contains a non-empty critical weighted independent set if, and only if, there is a vertex $v \in V(G)$ such that $\alpha_w(B(G)) = \alpha_w(B(G) - \{v, v'\} - N(\{v, v'\}) + 2w(v)$.*

Proof. This follows immediately from Theorem 5.5. □

Whether Theorem 4.2 is extendable to the weighted case is an open question. Is every critical weighted independent set contained in a maximum critical weighted independent set? The proof that every critical independent set is contained in a maximum critical independent set made use of Theorem 2.7, which cannot be extended: it is not true that, if I is a critical weighted independent set, then there is a matching from $N(I)$ to I. Consider the complete graph K_3 with three vertices. Let the vertices be $V = \{v_1, v_2, v_3\}$, having weights $w_1 = 1$, $w_2 = \frac{1}{2}$, and $w_3 = \frac{1}{2}$. Let $I = \{v_1\}$. It is easy to verify that I is a critical weighted independent set of K_3. But there is no matching from $N(I) = \{v_2, v_3\}$ to I.

6. Coda: Critical Independence Reductions for Fullerenes

This investigation was inspired by an attempt, using Butenko and Trukhanov's theorem and Ageev's algorithm, to reduce the problem of finding maximum independent sets in fullerene graphs—there is strong statistical evidence that the independence number of a fullerene is a predictor of its stability [6]. No reduction was found and, in fact, no reduction is possible.

Definition 6.1. A *fullerene* or *fullerene graph* is a connected, cubic, planar graph whose faces are either pentagons or hexagons.

Theorem 6.2. *The empty set is the only critical independent set in a fullerene. (Equivalently, for every non-empty independent set I of a fullerene, $|N(I)| > |I|$.)*

Proof. Suppose G is a fullerene and G contains a non-empty independent set I such that $|N(I)| \leq |I|$. Since G is cubic, there are $3|I|$ edges incident to set I. There are at least $3|I|$ edges incident to $N(I)$. Thus, $3|N(I)| \geq 3|I|$ and $|N(I)| \geq |I|$ (and $|N(I)| = |I|$). Thus the graph $G[I \cup N(I)]$, induced on $I \cup N(I)$, is bipartite. Since G is connected, $G = G[I \cup N(I)]$. So G is bipartite, which contradicts the fact that fullerenes are not bipartite (it is a simple consequence of Euler's Theorem that they have twelve pentagonal faces and, thus, odd cycles). □

The details of the proof actually give the following stronger theorem.

Theorem 6.3. *If G is a connected, non-bipartite, regular graph then the empty set is the only critical independent set.*

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