The Cvetkovic bound of a graph $G$ is the minimum of the number of non-negative and non-positive eigenvalues of the adjacency matrix of the graph.

**Notation:** Let $\rho_0$ be the number of 0 eigenvalues, $\rho_+$ be the number of positive eigenvalues, and $\rho_-$ be the number of negative eigenvalues. Then Cvetkovic’s Theorem says $\alpha \leq \rho_0 + \min\{\rho_+, \rho_-\}$.

1. Login to your Sage/Cocalc account.
   
   (a) Start the Chrome browser.
   (b) Go to http://cocalc.com and login.
   (c) Click on the Project for our class.
   (d) Click “New”, type s08 in the blank and click “Sage Worksheet”.

2. Here are functions we can use in an investigation of this useful bound:

   ```python
   def independence_number(g):
       return g.independent_set(value_only = True)
   
   def count_zero_eigenvalues(g):
       A = g.adjacency_matrix()
       Evals = A.eigenvalues()
       count = 0
       for lam in Evals:
           if lam == 0:
               count = count + 1
       return count
   ```

3. Now imitate this code and define your own functions to count the numbers of positive and negative eigenvalues of a graph adjacency matrix. Call them `count_positive_eigenvalues` and `count_negative_eigenvalues`.

4. Now we can code the Cvetkovic bound.

   ```python
   def cvetkovic_bound(g):
       p_0 = count_zero_eigenvalues(g)
       p_plus = count_positive_eigenvalues(g)
       p_minus = count_negative_eigenvalues(g)
       return p_0 + min(p_plus, p_minus)
   ```
5. Let’s find the Cvetkovic bound for $p_3$—and see all the details to check.

(a) Define $p_3$. Run `p3=graphs.PathGraph(3).show` to check the graph is what you think it is: `p3.show()`.

(b) We’ll find the adjacency matrix $A$. Run `A = p3.adjacency_matrix()`. Evaluate $A$ to see what that is.

(c) Then find the eigenvalues. Run `A.eigenvalues()`.

(d) Finally find our bound. Run: `cvetkovic_bound(p3)`.

6. Repeat the above steps for the cycle on 4 vertices, $c_4$. Start by defining your graph: `c4=graphs.CycleGraph(4)`.

7. Repeat the above steps for the Petersen Graph. Start by defining your graph: `pete=graphs.PetersenGraph()`.

8. No characterization of the graphs where the independence number equals the Cvetkovic bound is known. Let’s find some examples. To find all connected graphs on 4 vertices where these invariants are equal, run:

```python
for g in graphs.nauty_geng("4 -c"):
    if independence_number(g) == cvetkovic_bound(g):
        g.show()
        print g.graph6_string()
```

What did you get?

9. Repeat this to find all the graphs on 5 vertices where equality holds.

10. How many graphs on 6 vertices have equality? (Do you see anything in common—can you conjecture a characterization!?)