LARSON—MATH 756—NOTES 15
Linear Algebra—Cvetkovic Bound

1. The eigenvalues of a real symmetric (square) matrix A are real.

2. For any $n \times n$ real symmetric matrix A, $\mathbb{R}^n$ has a basis of orthogonal eigenvectors of A.

3. Rayleigh-Ritz Lemma: if the eigenvalues of a real symmetric matrix A are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, with corresponding orthonormal eigenvectors $\hat{u}_1, \ldots, \hat{u_n}$ and $W = \text{Span}\{\hat{u}_p, \ldots, \hat{u}_q\}$, with $1 \leq p \leq q \leq n$, then if $\hat{x} \in W$ then $\lambda_q \leq \hat{x} \cdot \hat{x} \leq \lambda_p$.

4. The **Rayleigh-Ritz Theorem** (Min-Max Theorem) says that if the eigenvalues of a real symmetric matrix A are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then $\lambda_n = \min \hat{x}^T A \hat{x}$ for all unit vectors $\hat{x}$, and $\lambda_1 = \max \hat{x}^T A \hat{x}$ for all unit vectors $\hat{x}$.

5. Lemma: If V is a vector space, and U and W are subspaces with $dim(U) + dim(W) > dim(V)$ then there is a non-zero vector $\hat{x} \in U \cap W$.

6. **Interlacing Theorem**. If A is a $n \times n$ symmetric matrix of the form $\begin{pmatrix} B & C \\ C^T & D \end{pmatrix}$, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, and B is a $m \times m$ symmetric matrix with eigenvalues $\mu_1 \geq \ldots \geq \mu_m$, then $\lambda_i \geq \mu_i$ for $i = 1, \ldots, m$.

   We'll prove the first half of the Interlacing Theorem in class, namely that $\mu_i \leq \lambda_i$, for each $i \in \{1, \ldots, m\}$.In the **homework** imitate this proof, applied to matrix $-A$, to **show** that $\lambda_{i+(n-m)} \leq \mu_i$, for each $i \in \{1, \ldots, m\}$.

7. **Cvetkovic's Theorem**: the independence number α of a graph is no more than the number of non-negative eigenvalues of the graph (and also no more than the number of non-positive eigenvalues).

   Notation: Let ρ_0 be the number of 0 eigenvalues, ρ_+ be the number of positive eigenvalues, and ρ_- be the number of negative eigenvalues. Then **Cvetkovic's Theorem** says $\alpha \leq \rho_0 + \min\{\rho_+, \rho_-\}$.

   In class we'll show the first part of the Interlacing Theorem implies the first half of Cvetkovic's Theorem: the independence number of a graph is no more than the number of non-negative eigenvalues, that is, $\alpha \leq \rho_0 + \rho_+$.

   In the **homework** you'll use the second half of the Interlacing Theorem to **show** the second half of Cvetkovic's Theorem: the independence number of a graph is no more than the number of non-positive eigenvalues, that is, $\alpha \leq \rho_0 + \rho_-$.

8. Here are three other classic graph eigenvalue results:

   (a) If a graph is r-regular then its largest eigenvalue is r.

   (b) If a graph is bipartite then its spectrum is symmetric with respect to the origin (or, in other words, if λ is an eigenvalue then so is $-\lambda$).

   (c) In a tree, $p_0 = n - 2\nu$, where n is the order and ν is the matching number.