1. The eigenvalues of a real symmetric (square) matrix $A$ are real.

2. For any $n \times n$ real symmetric matrix $A$, $\mathbb{R}^n$ has a basis of orthogonal eigenvectors of $A$.

3. Rayleigh-Ritz Lemma: if the eigenvalues of a real symmetric matrix $A$ are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, with corresponding orthonormal eigenvectors $\hat{u}_1, \ldots, \hat{u}_n$ and $W = \text{Span}\{\hat{u}_p, \ldots, \hat{u}_q\}$, with $1 \leq p \leq q \leq n$, then if $\hat{x} \in W$ then $\lambda_q \leq \hat{x} \cdot \hat{x} \leq \lambda_p$.

4. The Rayleigh-Ritz Theorem (Min-Max Theorem) says that if the eigenvalues of a real symmetric matrix $A$ are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then $\lambda_n = \min \hat{x}^T A \hat{x}$ for all unit vectors $\hat{x}$, and $\lambda_1 = \max \hat{x}^T A \hat{x}$ for all unit vectors $\hat{x}$.

5. Lemma: If $V$ is a vector space, and $U$ and $W$ are subspaces with $\dim(U) + \dim(W) > \dim(V)$ then there is a non-zero vector $\hat{x} \in U \cap W$.

6. Interlacing Theorem. If $A$ is a $n \times n$ symmetric matrix of the form $egin{pmatrix} B & C \\ C^T & D \end{pmatrix}$, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, and $B$ is a $m \times m$ symmetric matrix with eigenvalues $\mu_1 \geq \ldots \geq \mu_m$, then $\lambda_i \geq \mu_i$ for $i = 1, \ldots, m$.

We’ll prove the first half of the Interlacing Theorem in class, namely that $\mu_i \leq \lambda_i$, for each $i \in \{1, \ldots, m\}$. In the homework imitate this proof, applied to matrix $-A$, to show that $\lambda_{i+(n-m)} \leq \mu_i$, for each $i \in \{1, \ldots, m\}$.

7. Cvetkovic’s Theorem: the independence number $\alpha$ of a graph is no more than the number of non-negative eigenvalues of the graph (and also no more than the number of non-positive eigenvalues).

Notation: Let $\rho_0$ be the number of 0 eigenvalues, $\rho_+$ be the number of positive eigenvalues, and $\rho_-$ be the number of negative eigenvalues. Then Cvetkovic’s Theorem says $\alpha \leq \rho_0 + \min\{\rho_+, \rho_-\}$.

In class we’ll show the first part of the Interlacing Theorem implies the first half of Cvetkovic’s Theorem: the independence number of a graph is no more than the number of non-negative eigenvalues, that is, $\alpha \leq \rho_0 + \rho_+$.

In the homework you’ll use the second half of the Interlacing Theorem to show the second half of Cvetkovic’s Theorem: the independence number of a graph is no more than the number of non-positive eigenvalues, that is, $\alpha \leq \rho_0 + \rho_-$.

8. Here are three other classic graph eigenvalue results:

(a) If a graph is $r$-regular then its largest eigenvalue is $r$.

(b) If a graph is bipartite then its spectrum is symmetric with respect to the origin (or, in other words, if $\lambda$ is an eigenvalue then so is $-\lambda$).

(c) In a tree, $\rho_0 = n - 2\nu$, where $n$ is the order and $\nu$ is the matching number.