LARSON—MATH 756—NOTES 12
LP—Critical Independence Theory

1. **Independence Number IP**. For a graph G with vertices $\{v_i\}$, and associated variables $\{x_i\}$, the integer program has:
   (a) An *objective function*: $\max \sum x_i$
   (b) *edge constraints*: $x_i + x_j \leq 1$, for each edge $v_i v_j$.
   (c) *binary or integer constraints*: $x_i \in \{0, 1\}$.

2. **Fractional Independence LP**. For a graph G with vertices $\{v_i\}$, and associated variables $\{x_i\}$, the linear program is the same except the binary constraints are replaced with *non-negative constraints*: $x_i \geq 0$.

3. The *fractional independence number* α_f is the optimum value of this "relaxed IP". It can be argued that $\alpha \leq \alpha_f$, and that $\alpha_f \geq \frac{n}{2}$.

4. **Balinski's Lemma**. For a graph G and fractional independence number α_f there is a feasible solution attaining this optimum value using only the numbers $\{1, 0, \frac{1}{2}\}$. **Notation**: If $\{x_i\}$ is an optimal $\{1, 0, \frac{1}{2}\}$ solution let $\mathcal{O} = \{v_i : x_i = 1\}$ (the "ones" set), $Z = \{v_i : x_i = 0\}$ (the "zeros" set), and $H = \{v_i : x_i = \frac{1}{2}\}$ (the "halfs" set). **Note:** $Z = N(\mathcal{O})$.

5. **Matching Lemma**: For any optimal $\{1, 0, \frac{1}{2}\}$ solution there is a matching that saturates $N(\mathcal{O})$ in $G[\mathcal{O} \cup N(\mathcal{O})]$, and $G[\mathcal{O} \cup N(\mathcal{O})]$ is KE.

6. **Nemhauser-Trotter Theorem**. If G is a graph with fractional independence number α_f and corresponding $\{1, 0, \frac{1}{2}\}$-solution $\{x_i\}$ then the set of "ones" $\mathcal{O}$ can be extended to a maximum independent set of G.

7. **Half-set Lemma**: Any non-empty independent set I in $G[H]$ "has more neighbors", that is, $|N(I)| > |I|$. (In our vocabulary $G[H]$ is *independence irreducible*).

8. **Picard-Queyranne Theorem**. If G is a graph with fractional independence number α_f and corresponding $\{1, 0, \frac{1}{2}\}$-solution $X = \{x_i\}$ with a maximum number of ones ($\mathcal{O}_X$ is maximum), and $Y = \{y_i\}$ is any other optimal solution with a maximum number of ones (so $\mathcal{O}_Y$ is also maximum) then $\mathcal{O}_X \cup N(\mathcal{O}_X) = \mathcal{O}_Y \cup N(\mathcal{O}_Y)$.

9. **Translation Lemma**. I is a critical independent set if and only if there is an optimal $\{0, 1, \frac{1}{2}\}$ solution with $\mathcal{O} = I$.

10. **Independence Decomposition Theorem**. If I is a maximum critical independent set in a graph G and $X = I \cup N(I)$ and $X^c = V(G) \setminus X$ then:
    (a) $G[X]$ is KE.
    (b) $G[X^c]$ is independence irreducible,
    (c) $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$, and
    (d) if J is any other maximum critical independent set, then $X = J \cup N(J)$.

11. **Matching Decomposition Corollary**. If I is a maximum critical independent set in a graph G and $X = I \cup N(I)$ and $X^c = V(G) \setminus X$ then $\nu(G) = \nu(G[X]) + \nu(G[X^c])$.

12. **KE Characterization Theorem**. A graph is KE if and only if $\alpha = \alpha_f$.