LARSON—MATH 756—NOTES 10
Independence Decomposition Theorem

1. **Independence Number IP**. For a graph G with vertices $\{v_i\}$, and associated variables $\{x_i\}$, the integer program has:
   (a) An *objective function*: $\max \sum x_i$
   (b) *edge constraints*: $x_i + x_j \leq 1$, for each edge $v_i v_j$.
   (c) *binary or integer constraints*: $x_i \in \{0, 1\}$.
2. We can *prove*: α is the optimal value of this IP and that, for an optimal feasible solution $\{x_i\}$, the set of vertices corresponding to the "ones", namely $\{v_i : x_i = 1\}$, is a maximum independent set.
3. **Fractional Independence LP**. For a graph G with vertices $\{v_i\}$, and associated variables $\{x_i\}$, the linear program has:
   (a) An *objective function*: $\max \sum x_i$
   (b) *edge constraints*: $x_i + x_j \leq 1$, for each edge $v_i v_j$.
   (c) *non-negative constraints*: $x_i \geq 0$.
4. The *fractional independence number* α_f is the optimum value of this "relaxed IP". It can be argued that $\alpha \leq \alpha_f$, and that $\alpha_f \geq \frac{n}{2}$.
5. **Balinski's Lemma**. For a graph G and fractional independence number α_f there is a feasible solution attaining this optimum value using only the numbers $\{0, 1, \frac{1}{2}\}$.
6. **Nemhauser-Trotter Theorem**. If G is a graph with fractional independence number α_f and corresponding $\{1, 0, \frac{1}{2}\}$-solution $\{x_i\}$ then the set of "ones", namely $I = \{v_i : x_i = 1\}$ can be extended to a maximum independent set of G.
7. **KE Characterization Theorem**. A graph is KE if and only if $\alpha = \alpha_f$.
8. **Picard-Queyranne Theorem**. If G is a graph with fractional independence number α_f and corresponding $\{1, 0, \frac{1}{2}\}$-solution with a maximum number of integer (0-1) values, then any other optimal solution with a maximum number of integer solutions has the same set of integer solutions.
9. **Independence Decomposition Theorem**. If I is a maximum critical independent set in a graph G and $X = I \cup N(I)$ and $X^c = V(G) \setminus X$ then:
   (a) $G[X]$ is KE.
   (b) $G[X^c]$ is independence irreducible,
   (c) $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$, and
   (d) X is the unique set with these properties.