LARSON—MATH 756—NOTES 08
Independence Number IP—Fractional Independence LP

1. **Independence Number IP**. For a graph G with vertices $\{v_i\}$, and associated variables $\{x_i\}$, the integer program has:
   (a) An *objective function*: $\max \sum x_i$
   (b) *edge constraints*: $x_i + x_j \leq 1$, for each edge $v_i v_j$.
   (c) *binary or integer constraints*: $x_i \in \{0, 1\}$.
2. We can *prove*: α is the optimal value of this IP and that, for an optimal feasible solution $\{x_i\}$, the set of vertices corresponding to the "ones", namely $\{v_i : x_i = 1\}$, is a maximum independent set.
3. **Fractional Independence LP**. For a graph G with vertices $\{v_i\}$, and associated variables $\{x_i\}$, the linear program has:
   (a) An *objective function*: $\max \sum x_i$
   (b) *edge constraints*: $x_i + x_j \leq 1$, for each edge $v_i v_j$.
   (c) *non-negative constraints*: $x_i \geq 0$.
4. The *fractional independence number* α_f is the optimum value of this "relaxed IP". It can be argued that $\alpha \leq \alpha_f$, and that $\alpha_f \geq \frac{n}{2}$.
5. **Balinski's Lemma**. For a graph G and fractional independence number α_f there is a feasible solution attaining this optimum value using only the numbers $\{0, 1, \frac{1}{2}\}$.
6. **Nemhauser-Trotter Theorem**. If G is a graph with fractional independence number α_f and corresponding $\{1, 0, \frac{1}{2}\}$-solution $\{x_i\}$ then the set of "ones", namely $I = \{v_i : x_i = 1\}$ can be extended to a maximum independent set of G.
7. Related lemmas and results. Assume $\{x_i\}$ is an optimal solution to the fractional independence LP.
   (a) Let G' be the bipartite graph formed from $G[I \cup N(I)]$ by deleting all edges between vertices of $N(I)$. There is a matching that saturates $N(I)$ in G'.
   (b) There is a matching which saturates $N(I)$ in $G[I \cup N(I)]$.
   (c) The graph $G[I \cup N(I)]$ is König-Egervary.
8. **KE Characterization Theorem**. A graph is KE if and only if $\alpha = \alpha_f$.