LARSON—MATH 756—NOTES 05
Bipartite Graphs & Matching

1. **Bipartite Graphs**. A graph is *bipartite* if its vertex set V can be partitioned into independent sets X, Y. Clearly, if a graph is bipartite with bipartition (X, Y) then $\alpha \geq \max\{|X|, |Y|\}$.

   The theory of independent sets in bipartite graphs is well-developed and is connected to matching theory for bipartite graphs.

   (a) A *matching* in a graph is a set of independent edges, that is, edges which are pair-wise non-incident. If M is a matching then a vertex incident to some edge in M is *saturated* while a vertex not incident to an edge in M is *unsaturated*. An M-alternating path is a path whose edges alternate between edges in M and not in M. An M-augmenting path is a path which is M-alternating and whose initial and final vertices are unsaturated.

   (b) **Berge's Theorem**. A matching M in a graph is maximum if and only if there are no M-augmenting paths.

   (c) The matching number ν of a graph is the cardinality of a maximum matching.

   (d) **König's Theorem** (or the König-Egervary Theorem). For a bipartite graph, $\alpha + \nu = n$.

   (e) **König's Theorem (Mini-max version)** . For a bipartite graph, $\nu = \beta$ (where β is the cardinality of a minimum vertex cover, that is, a minimum cardinality set of vertices incident to every edge in the graph).

2. **König-Egervary Graphs** A graph is a *König-Egervary Graph* (or **KE graph)** if $\alpha + \nu = n$.