Bipartite Graphs & Matching

1. Bipartite Graphs. A graph is *bipartite* if its vertex set $V$ can be partitioned into independent sets $X, Y$. Clearly, if a graph is bipartite with bipartition $(X, Y)$ then $\alpha \geq \max\{|X|, |Y|\}$.

   The theory of independent sets in bipartite graphs is well-developed and is connected to matching theory for bipartite graphs.

   (a) A *matching* in a graph is a set of independent edges, that is, edges which are pair-wise non-incident. If $M$ is a matching then a vertex incident to some edge in $M$ is *saturated* while a vertex not incident to an edge in $M$ is *unsaturated*. An $M$-alternating path is a path whose edges alternate between edges in $M$ and not in $M$. An $M$-augmenting path is a path which is $M$-alternating and whose initial and final vertices are unsaturated.

   (b) Berge’s Theorem. A matching $M$ in a graph is maximum if and only if there are no $M$-augmenting paths.

   (c) The matching number $\nu$ of a graph is the cardinality of a maximum matching.

   (d) König’s Theorem (or the König-Egervary Theorem). For a bipartite graph, $\alpha + \nu = n$.

2. König-Egervary Graphs A graph is a *König-Egervary Graph* (or KE graph) if $\alpha + \nu = n$. 