LARSON—MATH 756—NOTES 02
Radius & Matching

1. **Notation**. We use $V = V(G)$ for the vertex set of a graph G and $E = E(G)$ for the edge set. The *order* n of the graph is the cardinality of V and the *size* m is the cardinality of the edge set.

   $v \sim w$ for vertex v is adjacent to vertex w or, equivalently, $vw \in E$; $N(v)$ is the set of *neighbors* of v, or vertices adjacent to v.

2. **Definition**. An *independent set* in a graph is a set of vertices which are pair-wise non-adjacent. A *maximum independent set* (MIS) is a largest cardinality independent set. The *independence number* α is the cardinality of an MIS.

3. **Independence Number Theory** consists of:

   - Bounds (upper and lower bounds for α).
   - Structure (subgraphs and classes with certain useful properties).
   - Algorithms (for finding maximum independent sets in general graphs, or special classes).

4. **Tarjan-Trojanowski**. For any graph G and any vertex v,

$$\alpha(G) = \max\{\alpha(G - v), \alpha(G - v - N(v)) + 1\}.$$

5. **The Induced Path Theorem**. A connected graph with radius r has an induced path with at least $2r - 1$ vertices. A corollary is that any graph has an independent set with at least r vertices, and $\alpha \geq r$.

6. **Bipartite Graphs**. A graph is *bipartite* if its vertex set V can be partitioned into independent sets X, Y. Clearly, if a graph is bipartite with bipartition (X, Y) then $\alpha \geq \max\{|X|, |Y|\}$.

   The theory of independent sets in bipartite graphs is well-developed and is connected to matching theory for bipartite graphs.

   (a) A *matching* in a graph is a set of independent edges, that is, edges which are pair-wise non-incident. If M is a matching then a vertex incident to some edge in M is *saturated* while a vertex not incident to an edge in M is *unsaturated*. An M-alternating path is a path whose edges alternate between edges in M and not in M. An M-augmenting path is a path which is M-alternating and whose initial and final vertices are unsaturated.

   (b) **Berge's Theorem**. A matching M in a graph is maximum if and only if there are no M-augmenting paths.

   (c) The matching number ν of a graph is the cardinality of a maximum matching.

   (d) **König's Theorem** (or the König-Egervary Theorem). For a bipartite graph, $\alpha + \nu = n$.