LARSON—MATH 756—NOTES 01
Independent Sets & Independence Number

Bondy & Murty's *Graph Theory with Applications* (1976) is a very good free book with many of the theorems I would expect you to have seen before. It is available at: `http://www.iro.umontreal.ca/~hahn/IFT3545/GTWA.pdf`. This book and Diestel's more recent book (with modern theory) are linked on my web page.

1. **Notation**. We use $V = V(G)$ for the vertex set of a graph G and $E = E(G)$ for the edge set. The *order* n of the graph is the cardinality of V and the *size* m is the cardinality of the edge set.

   $v \sim w$ for vertex v is adjacent to vertex w or, equivalently, $vw \in E$; $N(v)$ is the set of *neighbors* of v, or vertices adjacent to v.

2. **Definition**. An *independent set* in a graph is a set of vertices which are pair-wise non-adjacent. A *maximum independent set* (MIS) is a largest cardinality independent set. The *independence number* α is the cardinality of an MIS.

3. **Two Hard Problems**. Find an efficient algorithm for finding an MIS or α in a graph.

   While this is probably not possible mathematicians do research on two related problems which can be useful tools in practice:

   (a) Find classes of graphs where it is possible to find an MIS or α efficiently.

   (b) Find (upper or lower) *bounds* for α.

4. **Bipartite Graphs**. A graph is *bipartite* if its vertex set V can be partitioned into independent sets X, Y. Clearly, if a graph is bipartite with bipartition (X, Y) then $\alpha \geq \max\{|X|, |Y|\}$.

   The theory of independent sets in bipartite graphs is well-developed and is connected to matching theory for bipartite graphs.

   (a) A *matching* in a graph is a set of independent edges, that is, edges which are pair-wise non-incident. If M is a matching then a vertex incident to some edge in M is *saturated* while a vertex not incident to an edge in M is *unsaturated*. An M-alternating path is a path whose edges alternate between edges in M and not in M. An M-augmenting path is a path which is M-alternating and whose initial and final vertices are unsaturated.

   (b) **Berge's Theorem**. A matching M in a graph is maximum if and only if there are no M-augmenting paths.

   (c) The matching number ν of a graph is the cardinality of a maximum matching.

   (d) **König's Theorem** (or the König-Egervary Theorem). For a bipartite graph, $\alpha + \nu = n$.