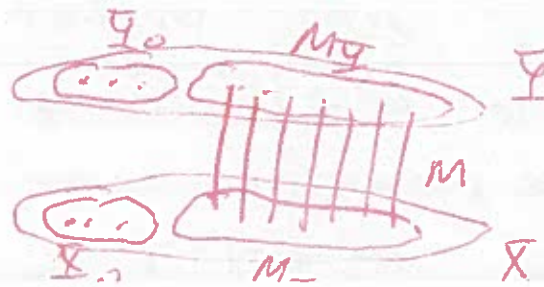
Königs Theorem

If G is a bipartite graph then the independence number α and the matching number ν sum to the order n:

$$\alpha + \nu = n$$

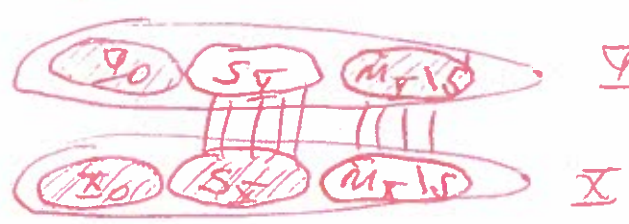
Proof

Let (X, Y) be a partition of $V(G)$ into independent sets, let M be a maximum matching (so $\nu = |M|$) and let M_X be the vertices in X that are incident to edges in M and M_Y be the vertices in Y incident to edges in M. Let $X_0 = X \setminus M_X$ and $Y_0 = Y \setminus M_Y$. Note that there are no edges from any vertex in X_0 to any vertex in Y_0



Let S' be the set of vertices that can be reached by M-alternating paths from vertices in Y_0. It is important to note that S' cannot contain any vertex in Y_0 – if it did there would be an M-augmenting path from some M-unsaturated vertex in Y_0 to an M-unsaturated vertex in Y_0 – but then Berge's Theorem implies that M is <u>not</u> a maximum matching. Let $S_X = S' \cap M_X$, $S_Y = S' \cap M_Y$ and $I = Y_0 \cup S'_X \cup (X \setminus S'_Y)$. Note that $Y \setminus S_Y$ includes every vertex in Y_0 as well as every vertex in $M_Y \setminus S'$. I is indep.

so $\alpha \geq |I| = n - \nu$



We showed $\alpha \leq n - \nu$,

<u>so</u> $\alpha = n - \nu$ _