Konig's theorem

If \( G \) is a bipartite graph then the independence number \( \alpha \) and the matching number \( \nu \) sum to the order \( n \):

\[ \alpha + \nu = n \]

Proof

Let \( (\overline{V}, V) \) be a partition of \( V(G) \) into independent sets, let \( M \) be a maximum matching (so \( \nu = |M| \)) and let \( M_{\overline{V}} \) be the vertices in \( \overline{V} \) that are incident to edges in \( M \) and \( M_V \) be the vertices in \( V \) incident to edges in \( M \). Let \( \overline{V}_0 = \overline{V} \setminus M_{\overline{V}} \) and \( V_0 = V \setminus M_V \). Note that there are no edges from any vertex in \( \overline{V}_0 \) to any vertex in \( V_0 \).
Let $S'$ be the set of vertices that can be reached by $m$-alternating paths from vertices in $X_0$. It is important to note that $S'$ cannot contain any vertex in $T_0$ if it did there would be an $m$-augmenting path from some $M$-unsaturated vertex in $T_0$ to an $M$-unsaturated vertex in $T_0$ but then Berge's Theorem implies that $M$ is not a maximum matching. Let $S^v = S' \cap M^v$, $S^v = S' \cap M^v$ and

$I = T_0 \cup S^v \cup (T \setminus S^v)$. Note that $T \setminus S^v$ includes every vertex in $T_0$ as well as every vertex in $M_0 \setminus S'$. $I$ is independent.

So $x \geq |I| = n - r$.

We showed $x \leq n - r$,

So $x = n - r$. 