Last name ______________________

First name ______________________

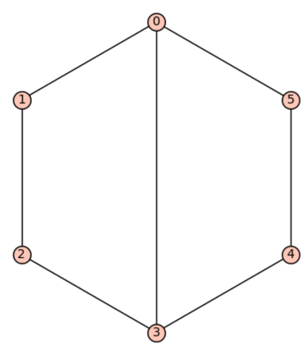
LARSON—MATH 756—HOMEWORK WORKSHEET 03

For a connected graph, the *distance* $d(v,w)$ from vertex v to vertex w is the length (number of edges) in a shortest path from v to w. The *eccentricity* $e(v)$ of a vertex v is the maximum distance to any other vertex; formally, this is $e(v) = \max\{d(v,w) : w \in V\}$. The *radius* r of a (connected) graph is the minimum eccentricity of any of its vertices.

The Induced Path Theorem. A connected graph with radius r has an induced path with at least $2r-1$ vertices. A corollary is that any graph has an independent set with at least r vertices, and $\alpha \geq r$.

A generalization of the Induced Path Theorem is that radius-critical graphs have a certain structure (forthcoming!). A graph is *radius-critical* if removing any non-cut vertex reduces its radius.

1. Find two non-isomorphic radius-critical subgraphs of the following graph. Check that each has an induced path with at least $2r-1$ vertices.



2. Show: **Tarjan-Trojanowski**. For any graph G and any vertex v,

$$\alpha(G) = \max\{\alpha(G-v), \alpha(G-v-N(v))+1\}.$$

3. **Lovasz' Theta**. Given a graph G with n vertices $\{v_1, \ldots, v_n\}$, an *orthonormal representation* is a collection of vectors $\{\vec{u_i}\} = \{\vec{u_1}. \ldots, \vec{u_n}\}$ such that vectors corresponding to any pair of non-adjacent vertices are orthogonal.

$$\vartheta(G) = \min_{\{\vec{u_i}\},\vec{c}} \max_i \frac{1}{(\vec{c}\cdot\vec{u_i})^2},$$

where the vector $\vec{c}$ in this definition is any *unit vector*. So the definition of ϑ of a graph is defined over *all* orthonormal representations $\{\vec{u_i}\} = \{\vec{u_1}. \ldots, \vec{u_n}\}$, and over *all* unit vectors $\vec{c}$. $\vec{c}$ is called the **handle** of the representation.

Lovasz' Theta Bound. For any graph, $\alpha \leq \vartheta$.

Write up a proof of Lovasz' Theta bound with enough detail that you could reproduce and explain the proof.

4. Find and prove a formula for $\vartheta(p_n)$ where p_n is the path on n vertices.

5. **König-Egervary Graphs** A graph is a *König-Egervary Graph* (or **KE graph)** if $\alpha + \nu = n$. Find an infinite class of graphs which are KE but **not** bipartite.