For a connected graph, the distance $d(v, w)$ from vertex $v$ to vertex $w$ is the length (number of edges) in a shortest path from $v$ to $w$. The eccentricity $e(v)$ of a vertex $v$ is the maximum distance to any other vertex; formally, this is $e(v) = \max\{d(v, w) : w \in V\}$. The radius $r$ of a (connected) graph is the minimum eccentricity of any of its vertices.

The Induced Path Theorem. A connected graph with radius $r$ has an induced path with at least $2r - 1$ vertices. A corollary is that any graph has an independent set with at least $r$ vertices, and $\alpha \geq r$.

A generalization of the Induced Path Theorem is that radius-critical graphs have a certain structure (forthcoming!). A graph is radius-critical if removing any non-cut vertex reduces its radius.

1. Find two non-isomorphic radius-critical subgraphs of the following graph. Check that each has an induced path with at least $2r - 1$ vertices.

2. Show: Tarjan-Trojanowski. For any graph $G$ and any vertex $v$,

$$\alpha(G) = \max\{\alpha(G - v), \alpha(G - v - N(v)) + 1\}.$$ 

3. Lovasz’ Theta. Given a graph $G$ with $n$ vertices $\{v_1, \ldots, v_n\}$, an orthonormal representation is a collection of vectors $\{\vec{u}_i\} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ such that vectors corresponding to any pair of non-adjacent vertices are orthogonal.

$$\varrho(G) = \min_{\{\vec{u}_i\}, \vec{c}} \max_i \frac{1}{(\vec{c} \cdot \vec{u}_i)^2},$$

where the vector $\vec{c}$ in this definition is any unit vector. So the definition of $\varrho$ of a graph is defined over all orthonormal representations $\{\vec{u}_i\} = \{\vec{u}_1, \ldots, \vec{u}_n\}$, and over all unit vectors $\vec{c}$. $\vec{c}$ is called the handle of the representation.
**Lovasz’ Theta Bound.** For any graph, $\alpha \leq \vartheta$.

Write up a proof of Lovasz’ Theta bound with enough detail that you could reproduce and explain the proof.

4. Find and prove a formula for $\vartheta(p_n)$ where $p_n$ is the path on $n$ vertices.

5. **König-Egervary Graphs** A graph is a *König-Egervary Graph* (or *KE graph*) if $\alpha + \nu = n$. Find an infinite class of graphs which are KE but not bipartite.