The independence number of a graph is closely related to many other graph invariants. We record a sample here. You should be ready to present proofs of at least some of these theorems in class next week. Prof Bushaw will take volunteers on Monday and then more volunteers on Wednesday. Each student should prove at least one theorem before any student is permitted to prove a second theorem. The goal is to prove all of these in class. A write-up will be due of all these theorems in the future. You may (and are encouraged) to work together on this assignment.

A clique in a graph is a complete subgraph. The clique number \( \omega \) of a graph is the cardinality of a maximum clique.

The complement \( \bar{G} \) of a graph \( G \) is the graph with vertex set \( V(G) \) where \( vw \in E(\bar{G}) \iff vw \notin E(G) \).

1. Show: \( \alpha(G) = \omega(\bar{G}) \).
2. Show: \( \alpha + \omega \leq n + 1 \).

For a connected graph \( G \), a subset \( D \) of the vertex set is dominating (or a dominating set) if every vertex not in \( D \) is adjacent to some (at least one) vertex in \( D \); if the graph is not connected a set \( D \) of vertices is dominating if it dominates each connected component of the graph: in other words, each vertex is either in \( D \) or is adjacent to some vertex in \( D \). The domination number \( \gamma \) of a graph is the cardinality of a minimum dominating set.

3. Show: \( \alpha \geq \gamma \).

The degree \( d(v) \) of a vertex \( v \) is the number of vertices \( v \) is adjacent to. The maximum degree \( \Delta \) is the largest degree; in other words, \( \Delta = \max\{d(v) : v \in V\} \). The minimum degree \( \delta \) is the smallest degree of the graph.

4. Show: \( \alpha \leq n - \delta \).
5. Show: \( \alpha \geq \frac{n}{\Delta + 1} \).
6. Show: \( \alpha \leq n - \frac{\varepsilon}{\Delta} \).

A proper coloring of a graph is a partition of the vertex set into independent subsets \( V_1, V_2, \ldots, V_k \) (these sets are called colors or color classes). The chromatic number \( \chi \) of a graph is the smallest number of sets in any proper coloring of the graph.

7. Show: \( \alpha \geq \frac{n}{\chi} \).
A clique cover of a graph is a partition of the vertex set into sets which induce cliques. The clique covering number \( \bar{\omega} \) is the number of sets in a clique cover with a minimum number of sets.

8. Show: \( \alpha \leq \bar{\omega} \).

The independence number is sandwiched between two functions of the matching number \( \nu \).

9. Show: \( \alpha \leq n - \nu \).

10. Show: \( \alpha \geq n - 2\nu \).

A covering of a graph is a subset \( C \) of the vertex set such that each edge of the graph is incident to at least one vertex in \( C \). The covering number \( \tau \) of a graph is the number of vertices of a minimum cardinality covering set.

11. Show: \( \alpha + \tau = n \).

For a connected graph, the distance \( d(v, w) \) from vertex \( v \) to vertex \( w \) is the length (number of edges) in a shortest path from \( v \) to \( w \). The eccentricity \( e(v) \) of a vertex \( v \) is the maximum distance to any other vertex; formally, this is \( e(v) = \max\{d(v, w) : w \in V\} \). The radius \( r \) of a (connected) graph is the minimum eccentricity of any of its vertices.

12. **Bonus.** Show: the length of a maximum induced path in a graph is at least \( 2r - 1 \). Then show the corollary: \( \alpha \geq r \).