Last name ______________________

First name ______________________

LARSON—MATH 756—HOMEWORK WORKSHEET 01
Bounds for the Independence Number of a Graph

The independence number of a graph is closely related to many other graph invariants. We record a sample here.

You should be ready to present proofs of at least some of these theorems in class next week. Prof Bushaw will take volunteers on Monday and then more volunteers on Wednesday. Each student should prove at least one theorem before any student is permitted to prove a second theorem. The goal is to prove all of these in class. A write-up will be due of all these theorems in the future. You *may* (and are encouraged) to work together on this assignment. .

A *clique* in a graph is a complete subgraph. The *clique number* of a graph ω is the cardinality of a maximum clique.

The complement $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ where $vw \in E(\bar{G}) \Leftrightarrow vw \notin E(G)$.

1. Show: $\alpha(G) = \omega(\bar{G})$.
2. Show: $\alpha + \omega \leq n + 1$.

For a connected graph G, a subset D of the vertex set is *dominating* (or a *dominating set*) if every vertex not in D is adjacent to *some* (at least one) vertex in D; if the graph is not connected a set D of vertices is dominating if it dominates each connected component of the graph: in other words, each vertex is either in D or is adjacent to some vertex in D. The *domination number* γ of a graph is the cardinality of a minimum dominating set.

3. Show: $\alpha \geq \gamma$.

The *degree* $d(v)$ of a vertex v is the number of vertices v is adjacent to. The maximum degree Δ is the largest degree; in other words, $\Delta = \max\{d(v) : v \in V\}$. The minimum degree δ is the smallest degree of the graph.

4. Show: $\alpha \leq n - \delta$.
5. Show : $\alpha \geq \frac{n}{\Delta+1}$.
6. Show: $\alpha \leq n - \frac{e}{\Delta}$.

A *proper coloring* of a graph is a partition of the vertex set into independent subsets $V_1, V_2, \ldots, V_k$ (these sets are called *colors* or *color classes*). The *chromatic number* χ of a graph is the smallest number of sets in any proper coloring of the graph.

7. Show: $\alpha \geq \frac{n}{\chi}$.

A *clique cover* of a graph is a partition of the vertex set into sets which induce cliques. The *clique covering number* $\bar{\omega}$ is the number of sets in a clique cover with a minimum number of sets.

8. Show: $\alpha \leq \bar{\omega}$.

The independence number is sandwiched between two functions of the matching number ν.

9. Show: $\alpha \leq n - \nu$.

10. Show: $\alpha \geq n - 2\nu$.

A *covering* of a graph is a subset C of the vertex set such that each edge of the graph is incident to at least one vertex in C. The *covering number* τ of a graph is the number of vertices of a minimum cardinality covering set.

11. Show: $\alpha + \tau = n$.

For a connected graph, the *distance* $d(v, w)$ from vertex v to vertex w is the length (number of edges) in a shortest path from v to w. The *eccentricity* $e(v)$ of a vertex v is the maximum distance to any other vertex; formally, this is $e(v) = \max\{d(v, w) : w \in V\}$. The *radius* r of a (connected) graph is the minimum eccentricity of any of its vertices.

12. **Bonus**. Show: the length of a maximum induced path in a graph is at least $2r - 1$. Then show the corollary: $\alpha \geq r$.