Let G be a graph with vertices $V = \{v_i\}$. Associate a variable x_i to each vertex.

The Fractional Independence LP is:

$$\begin{cases} \max \sum x_i & \leftarrow \text{objective function} \\ x_i + x_j \leq 1, \text{ for each edge } v_i v_j & \text{(edge constraints)} \\ x_i \geq 0 & \leftarrow \text{Non-negativity constraints} \end{cases}$$

The fractional independence number α_f is defined to equal the optimum value of this linear program.

Balinski's Lemma There is an optimum feasible solution $\{x_i\}$ with $\alpha_f = \sum x_i$ and each $x_i \in \{0, 1, \frac{1}{2}\}$

Proof. Let $\{x_i\}$ be an optimum solution with a maximum number of $\{0, 1, \frac{1}{2}\}$ values

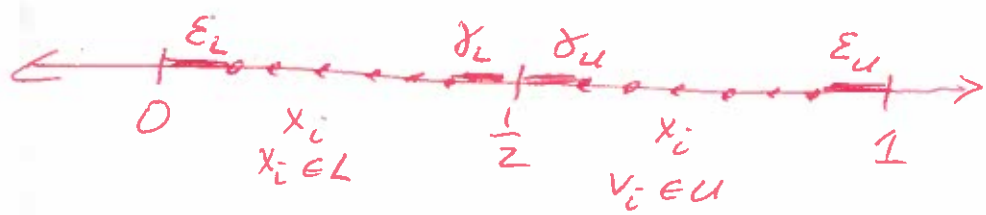
Let $O = \{v_i : x_i = 1\}$ $\quad L = \{v_i : 0 < x_i < \frac{1}{2}\}$

$Z = \{v_i : x_i = 0\}$ $\quad U = \{v_i : \frac{1}{2} < x_i < 1\}$

$H = \{v_i : x_i = \frac{1}{2}\}$ So, $\{O, Z, H, L, U\}$ is a partition of V

Now we will show that, unless L and U are empty, the solution $\{x_i\}$ can be tweaked to increase the number of $\{0, 1, \frac{1}{2}\}$ values. Since $\{x_i\}$ has a maximum number of these, it follows that L and U must be empty.

Here is a schematic of the values and quantities we'll define



Let $\varepsilon_L = \min\{x_i : x_i \in L\}$
$\varepsilon_U = \min\{1 - x_i : x_i \in U\}$
$\varepsilon = \min\{\varepsilon_L, \varepsilon_U\}$

$\gamma_U = \min\{x_i - \frac{1}{2} : x_i \in U\}$
$\gamma_L = \min\{\frac{1}{2} - x_i : x_i \in L\}$
$\gamma = \min\{\gamma_U, \gamma_L\}$

Case I. $|U| \geq |L|$. We increase all weights in U by ε, and decrease all weights in L by ε. All edge constraints remain satisfied and the objective is at least as large. Formally, let

$$x_i' = \begin{cases} x_i + \varepsilon & \text{if } v_i \in U \\ x_i - \varepsilon & \text{if } v_i \in L \\ x_i & \text{else} \end{cases}$$

Then $\{x_i'\}$ is feasible and

$$\alpha_f' = \sum x_i' \geq \sum x_i + |U|\varepsilon - |L|\varepsilon \geq \alpha_f$$

Case II. $|L| \leq |U|$. Similar, use γ. $\square$