Let $G$ be a graph with vertices $V = \{v_i \mid i \in \mathbb{N}\}$. Associate a variable $x_i$ to each vertex.

The Fractional Independence LP is:

$$\text{max } \sum x_i \quad \text{subject to:}$$

$$x_i + x_j \leq 1, \text{ for each edge } v_i, v_j \text{ (edge constraints)}$$

$$x_i \geq 0 \quad \text{non-negativity constraints}$$

The fractional independence number of $G$ is defined to equal the optimum value of this linear program.

**Balinski’s Lemma** There is an optimum feasible solution of the $\sum x_i$ with $\phi = \sum x_i$ and each $x_i \in \{0, 1/2\}$

**Proof.** Let $\sum x_i$ be an optimum solution with a maximum number of $\{0, 1/2\}$ values

Let $G = \{v_i : x_i = 1\}$

$L = \{v_i : 0 < x_i < 1/2\}$

$Z = \{v_i : x_i = 0\}$

$U = \{v_i : 1/2 < x_i < 1\}$

$H = \{v_i : x_i = 1/2\}$

So, $\{G, Z, H, L, U\}$ is a partition of $V$
Now we will show that, unless $L$ and $U$ are empty, the solution $\{x_i'\}$ can be tweaked to increase the number of $\xi, 1, \frac{1}{2}$ values. Since $\{x_i\}$ has a maximum number of these, it follows that $L$ and $U$ must be empty.

Here is a schematic of the values and quantities we'll define:

\[\begin{array}{cccccccc}
\varepsilon_L & \text{---} & \delta_L & \text{---} & \delta_U & \text{---} & \varepsilon_U \\
0 & x_i & \frac{1}{2} & x_i & v_i & U
\end{array}\]

Let 
\[\varepsilon_L = \min \{x_i : x_i \in L\}\]
\[\varepsilon_U = \min \{1-x_i : x_i \in U\}\]
\[\varepsilon = \min \{\varepsilon_L, \varepsilon_U\}\]
\[\delta_L = \min \{x_i - \frac{1}{2} : x_i \in L\}\]
\[\delta_U = \min \{\frac{1}{2} - x_i : x_i \in U\}\]
\[\gamma = \min \{\delta_U, \delta_L\}\]

**Case I.** $|U| \geq |L|$. We increase all weights in $U$ by $\varepsilon$, and decrease all weights in $L$ by $\varepsilon$. All edge constraints remain satisfied and the objective is at least as large. Formally, let 
\[x_i' = x_i + \varepsilon \text{ if } v_i \in U\]
\[x_i' = x_i - \varepsilon \text{ if } v_i \in L\]
\[x_i' = x_i \text{ else}\]
Then $\{x_i'\}$ is feasible and
\[\alpha' = \sum x_i' \geq \sum x_i + 1|U|\varepsilon - 1|L|\varepsilon \geq \alpha\]

**Case II.** $|L| \leq |U|$. Similar, use $\delta$. $\square$