Our goal is to compute the mobius function for the collection of subspaces of a vector space over a finite field, with the subspace relation. Our notation follows that of the Bender & Goldman article. Here is a small example to motivate their discussion.

Let \( V_n(q) \) be the vector space consisting of vectors with \( n \) components and entries from the field \( \mathbb{F}_q \).

Let \( L(V_n(q)) \) be the collection of subspaces of \( V_n(q) \).

For \( U, V \in L(V_n(q)) \) define \( U \leq V \) iff \( U \) is a subspace of \( V \). We claim that \((L(V_n(q)), \leq)\) is a poset.

Let \( \binom{n}{k}_q \) be the number of \( k \)-dimensional subspaces of \( V_n(q) \).

For any poset \( \mathbb{P} = (X, \leq) \) and elements \( x, y \in X \), let \([x, y] = \{ z : x \leq z \leq y \} \). \([x, y]\) is called an interval (by analogy with intervals on lines). And of course \(([x, y], \leq)\) is an induced poset of \( \mathbb{P} \).

An important and useful idea here, that we proved, is that all \( n \)-dimensional subspaces over the same field as isomorphic (and while being isomorphic is a technical mathematical term you should think of them as being really the same).

This isn’t a course on abstract or linear algebra—but the connections are many and we need to be facile with basic facts from these subjects. So first reload more linear algebra fundamentals.

1. Let \( U \) and \( V \) be vector spaces with \( U \) is a subspace of \( V \) (its not important to specify the field here—that they are over the same field comes from the definitions). Let \( \{\hat{u}_1, \hat{u}_2, \ldots, u_{\hat{u}(V)}\} \). Show that this basis can be extended to a basis for \( V \).
We will also show that for subspaces $U, V \in V_n(q)$ that the interval $[U, V]$ is isomorphic to the interval $[0, V/U]$. Let's see how this plays out with a small example.

2. Let $U = \{a \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} : a \in \mathbb{F}_3\}$. Write out $U$ explicitly. Find $d(U)$.

3. Let $V = \{a \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} : a, b \in \mathbb{F}_3\}$. Write out $V$ explicitly. Find $d(V)$.

Check that $U$ is a subspace of $V$.

4. Let $[v] = \{w \in V : v - w \in U\}$. Find $[v]$ for each $v \in V$.


6. $V/U$ can be viewed as a vector space with an analogous addition and scalar multiplication. Define an appropriate addition and scalar multiplication and show that $V/U$ is a vector space with this addition and scalar multiplication.

7. What is the “0” of the vector space $V/U$? (Show this.)

8. Which elements of $V/U$ are additive inverses? (Show).

9. Show that $V/U$ is isomorphic to $V_{d(V) - d(U)}(3)$.


11. Find $[0, V/U]$ (where 0 denotes the unique 0-dimensional subspace over $\mathbb{F}_3$).

12. Show that the poset $([U, V], \leq)$ is isomorphic to the poset $([0, V/U], \leq)$. You can do this with an explicit construction of a map $\phi : [U, V] \to [0, V/U]$—but you might also think about a general idea that will work in the abstract case.

Now, if you want to get ahead, show that for any subspaces $U, V \in V_n(q)$ that the interval $[U, V]$ is isomorphic to the interval $[0, V/U]$. 