Last name ______________________

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LARSON—MATH 750—HOMEWORK WORKSHEET 05
Delta, Zeta and Mobius Functions

For the poset $\mathbb{P} = (X, \leq)$, let $\mathcal{F} = \{f : X \times X \to \mathbb{R}\}$. Then define the *Kronecker delta function*:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and the *zeta function*:

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases}$$

So $\delta, \zeta \in \mathcal{F}$.

For functions $f, g \in \mathcal{F}$ the *convolution product* $f * g \in \mathcal{F}$ is:

$$\begin{cases} 0 & \text{if } x \not\leq y \\ \sum_{\{z : x \leq z \leq y\}} f(x,z)g(z,y) & \text{else} \end{cases}$$

Let $\mathcal{F}' = \{f \in \mathcal{F} : \forall y \in X \;\; f(y,y) \neq 0\}$. This will be the *group of invertible functions* on $\mathbb{P}$.

For $f \in \mathcal{F}'$ define g as follows:

$$g(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{f(y,y)} & \text{if } x = y \\ -\frac{1}{f(y,y)} \sum_{\{z : x \leq z < y\}} g(x,z)f(z,y) & \text{else} \end{cases}$$

We showed: for every $f \in \mathcal{F}'$, g is the inverse (with respect to the convolution product) of f, that is $f * g = \delta$ and $g * f = \delta$.

Since $\zeta \in \mathcal{F}'$ and thus ζ has an inverse, we *define* the *mobius function* μ to be this inverse; that is, μ is the function such that $\zeta * \mu = \mu * \zeta = \delta$.

Let $F : X \to \mathbb{R}$, and define $G : X \to \mathbb{R}$ as follows: $G(x) = \sum_{\{y : y \leq x\}} F(y)$.

Mobius Inversion Theorem:

$$F(x) = \sum_{\{y : y \leq x\}} \mu(y,x) \cdot G(y).$$

1. Is g as defined above well-defined? That is, will g produce a unique value for each $x \in X$ (and will it in fact always produce some value? It seems like it will, but the definition is recursive - and maybe convoluted). So it is worth addressing this.

   Hint: g is certainly well-defined for posets where the ground set X consists of a single element.

2. Is g unique as a inverse of $f \in \mathcal{F}'$? Can you argue that if $h : X \times X \to \mathbb{R}$ has the property that $h * f = f * h = \delta$ that $h = g$?

3. Let $\mathbb{P} = (X, \leq)$ be a poset where every pair of elements in X is comparable. Suppose $|X| = n$. Show that there is a labelling of the elements of X so that $x_1 < x_2 < \ldots < x_n$ (the issue here isn't the labelling but rather showing that $\mathbb{P}$ is *linear*).

4. Let $\mathbb{D} = ([10], |)$. Make a table with the elements in $[10]$ as the indices for the rows x and columns y with table entries $\mu(x, y)$. Use the definition of μ as the inverse of ζ to compute these values.