For the poset $\mathbb{P} = (X, \leq)$, let $\mathcal{F} = \{ f : X \times X \to \mathbb{R} \}$. Then define the Kronecker delta function:

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and the zeta function:

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases}$$

So $\delta, \zeta \in \mathcal{F}$.

For functions $f, g \in \mathcal{F}$ the convolution product $f * g \in \mathcal{F}$ is:

$$f * g = \begin{cases} 0 & \text{if } x \not\leq y \\ \sum_{\{z : x \leq z \leq y\}} f(x, z)g(z, y) & \text{else} \end{cases}$$

Let $\mathcal{F}' = \{ f \in \mathcal{F} : \forall y \in X \ f(y, y) \neq 0 \}$. This will be the group of invertible functions on $\mathbb{P}$.

For $f \in \mathcal{F}'$ define $g$ as follows:

$$g(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{f(y, y)} & \text{if } x = y \\ -\frac{1}{f(y, y)} \sum_{\{z : x \leq z < y\}} g(z, z) f(x, z) & \text{else} \end{cases}$$

We showed: for every $f \in \mathcal{F}'$, $g$ is the inverse (with respect to the convolution product) of $f$, that is $f * g = \delta$ and $g * f = \delta$.

Since $\zeta \in \mathcal{F}'$ and thus $\zeta$ has an inverse, we define the mobius function $\mu$ to be this inverse; that is, $\mu$ is the function such that $\zeta * \mu = \mu * \zeta = \delta$.

Let $F : X \to \mathbb{R}$, and define $G : X \to \mathbb{R}$ as follows: $G(x) = \sum_{\{y : y \leq x\}} F(y)$.

Mobius Inversion Theorem:

$$F(x) = \sum_{\{y : y \leq x\}} \mu(y, x) \cdot G(y).$$
1. Is $g$ as defined above well-defined? That is, will $g$ produce a unique value for each $x \in X$ (and will it in fact always produce some value? It seems like it will, but the definition is recursive - and maybe convoluted). So it is worth addressing this.

**Hint:** $g$ is certainly well-defined for posets where the ground set $X$ consists of a single element.

2. Is $g$ unique as an inverse of $f \in \mathcal{F}'$? Can you argue that if $h : X \times X \to \mathbb{R}$ has the property that $h \ast f = f \ast h = \delta$ that $h = g$?

3. Let $P = (X, \leq)$ be a poset where every pair of elements in $X$ is comparable. Suppose $|X| = n$. Show that there is a labelling of the elements of $X$ so that $x_1 < x_2 < \ldots < x_n$ (the issue here isn’t the labelling but rather showing that $P$ is linear).

4. Let $D = ([10], |)$. Make a table with the elements in $[10]$ as the indices for the rows $x$ and columns $y$ with table entries $\mu(x, y)$. Use the definition of $\mu$ as the inverse of $\zeta$ to compute these values.