For the poset \( P = (X, \leq) \), let \( \mathcal{F} = \{ f : X \times X \to \mathbb{R} \} \). Then define the Kronecker delta function:

\[
\delta(x, y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases}
\]

and the zeta function:

\[
\zeta(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
0 & \text{if } x \not\leq y
\end{cases}
\]

So \( \delta, \zeta \in \mathcal{F} \).

For functions \( f, g \in \mathcal{F} \) the convolution product \( f \ast g \in \mathcal{F} \) is:

\[
\begin{cases} 
0 & \text{if } x \not\leq y \\
\sum_{\{z : x \leq z \leq y\}} f(x, z)g(z, y) & \text{else}
\end{cases}
\]

1. Show that for any functions \( f, g, h \in \mathcal{F} \) \((f \ast g) \ast h = f \ast (g \ast h)\).

Now let \( \mathcal{F}' = \{ f \in \mathcal{F} : \forall y \in X \ f(y, y) \neq 0 \} \). This will be the group of invertible functions on \( P \).

For \( f \in \mathcal{F}' \) define \( g \) as follows:

\[
g(x, y) = \begin{cases} 
0 & \text{if } x \neq y \\
\frac{1}{f(y, y)} & \text{if } x = y \\
\frac{1}{f(y, y)} \sum_{\{z : x \leq z \leq y\}} g(x, z)f(z, y) & \text{else}
\end{cases}
\]

2. In class we showed that, for \( f \in \mathcal{F}' \) and \( g \) defined as above that \( g \) is a left inverse of \( f \), that is, that \( g \ast f = \delta \). Is \( g \) a right-inverse of \( f \), that is, is it true that \( f \ast g = \delta \)?

For any poset \( P \) define \( \mu \in \mathcal{F}' \) to be the inverse of the \( \zeta \) function for \( P \).

3. Let \( p_3 \) be the path on 3 vertices. Let \( X \) be the collection of connected non-trivial subgraphs of \( p_3 \) and let \( P = (X, \leq) \) be the graph poset where the relation “\( \leq \)” is defined to be the subgraph relation (here “subgraph” and “induced subgraph” is coincidentally the same). Make tables of values for the \( \zeta \) and \( \mu \) functions.

4. Let \( p_4 \) be the path on 4 vertices. Let \( X \) be the collection of connected non-trivial subgraphs of \( p_4 \) and let \( P = (X, \leq) \) be the graph poset where the relation “\( \leq \)” is defined to be the subgraph relation. Make tables of values for the \( \zeta \) and \( \mu \) functions.

5. Can you find an explicit formula for \( \mu \) in either of the last 2 examples? Can you find a formula for the general case: the poset of connected subgraphs of \( p_n \)?