Last name ______________________

First name ______________________

LARSON—MATH 750—HOMEWORK WORKSHEET 04
Delta, Zeta and Mobius Functions

For the poset $\mathbb{P} = (X, \leq)$, let $\mathcal{F} = \{f : X \times X \to \mathbb{R}\}$. Then define the *Kronecker delta function*:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and the *zeta function*:

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases}$$

So $\delta, \zeta \in \mathcal{F}$.

For functions $f, g \in \mathcal{F}$ the *convolution product* $f * g \in \mathcal{F}$ is:

$$\begin{cases} 0 & \text{if } x \not\leq y \\ \sum_{\{z : x \leq z \leq y\}} f(x,z)g(z,y) & \text{else} \end{cases}$$

1. Show that for any functions $f, g, h \in \mathcal{F}$ $(f * g) * h = f * (g * h)$.

   Now let $\mathcal{F}' = \{f \in \mathcal{F} : \forall y \in X \; f(y,y) \neq 0\}$. This will be the *group of invertible functions* on $\mathbb{P}$.

   For $f \in \mathcal{F}'$ define g as follows:

$$g(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{f(y,y)} & \text{if } x = y \\ -\frac{1}{f(y,y)} \sum_{\{z : x \leq z < y\}} g(x,z)f(z,y) & \text{else} \end{cases}$$

2. In class we showed that, for $f \in \mathcal{F}'$ and g defined as above that g is a *left inverse* of f, that is, that $g * f = \delta$. Is g a right-inverse of f, that is, is it true that $f * g = \delta$?

   For any poset $\mathbb{P}$ define $\mu \in \mathcal{F}'$ to be the inverse of the ζ function for $\mathbb{P}$.

3. Let p_3 be the path on 3 vertices. Let X be the collection of connected non-trivial subgraphs of p_3 and let $\mathbb{P} = (X, \leq)$ be the graph poset where the relation "$\leq$" is defined to be the subgraph relation (here "subgraph" and "induced subgraph" is coincidentally the same). Make tables of values for the ζ and μ functions.

4. Let p_4 be the path on 4 vertices. Let X be the collection of connected non-trivial subgraphs of p_4 and let $\mathbb{P} = (X, \leq)$ be the graph poset where the relation "$\leq$" is defined to be the subgraph relation . Make tables of values for the ζ and μ functions.

5. Can you find an explicit formula for μ in either of the last 2 examples? Can you find a formula for the general case: the poset of connected subgraphs of p_n?