LARSON—MATH 750—CLASSROOM WORKSHEET 24
Poset of Connected Subgraphs

Agenda

1. Prove the mobius function for the poset of connected subgraphs of a graph
2. Develop the poset of the eulerian subgraphs of a graph and its mobius function

Poset of Connected Subgraphs of a Graph

Let $G$ be a connected graph. Let $\mathcal{P}$ be the poset of connected subgraphs with the subgraph relation. We’ll show that the mobius function $\mu$ is given by:

$$\mu(H, G) = (-1)^{e(G) - e(H)},$$

when $H$ is a subgraph of $G$, and every edge of $H$ is incident to a vertex of $H$—and 0 otherwise where $e(H)$ is the size of $H$. (In particular if $H'$ contains an edge that is “independent” of $H$ then $\mu(H, H') = 0$).
We should always start by checking claims for small graphs.

1. Check this claim when $G$ is $k_3$ and $H$ is a single edge of that graph.
2. Check this claim when $G$ is $c_4$ and $H$ is a single edge of that graph.
3. Induction has worked for us before proving mobius function results. What graph invariant might seem very natural to induct on here?

4. Show that the claim is true for “small” graphs.
5. Write out $\mu(H, G)$ according to the theory we’ve developed. What are the two natural cases to consider here?
6. Rethink what it means to be a graph $H'$ between $H$ and $G$ and our induction hypothesis and see what that buys us.