Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 18
Inclusion-Exclusion

Given subsets $A_1, \ldots, A_n$ of a set S (with $\bar{A}_i = S \setminus A_i$), our goal is to find a formula for $|A_1 \cup \ldots \cup A_n|$.

We noted that $|\overline{A_1 \cup \ldots \cup A_n}| = |S| - |A_1 \cup \ldots \cup A_n|$ and $\overline{A_1 \cup \ldots \cup A_n} = \bar{A}_1 \cap \ldots \cap \bar{A}_n$. Thus to find $|A_1 \cup \ldots \cup A_n|$ it is enough to find $|\bar{A}_1 \cap \ldots \cap \bar{A}_n|$

Notation: We let $[n] = \{1, 2, \ldots, n\}$ and $\cup_{i \in [n]} A_i = A_1 \cup \ldots \cup A_n$.

We can now follow the same steps as our last (concrete) example to show that:

$$|S| - |\cup_{i \in [n]} A_i| = |\cap_{i \in [n]} \bar{A}_i| = \sum_{J \subseteq [n]} (-1)^{|J|} |\cap_{j \in J} A_j|.$$

For every $K \subseteq [n]$ (with $\bar{K} = [n] \setminus K$) we define:

$$S_K = \{s \in S : s \in \cap_{i \in \bar{K}} A_i \text{ and } s \in \cap_{j \in K} \bar{A}_j\}$$

Then we let $F(K) = |S_K|$, and $G(K) = \sum_{L \subseteq K} F(L)$.

From the Mobius Inversion Theorem we know $F(K) = \sum_{L \subseteq K} \cdot \mu(L, K) G(L)$.

We will now use the following 2 facts to get our result:

1. $\mu(L, K) = (-1)^{|K| - |L|}$ (the mobius function we proved for families of sets with inclusion), and
2. $G(L) = |\cap_{i \in \overline{L}} A_i|$ (proved below)

That's the set-up. Now let's do the work!

1. First let's knock down the main result using the above facts. This will imitate what we did in the particular example last class.

Now we'll show $G(L) = |\cap_{i \in \overline{L}} A_i|$. The main idea is to show that the non-empty S_M ($M \subseteq L$) sets are a partition of $\cap_{i \in \overline{L}} A_i$.

2. First let's see how the partition gives us the result.

Now let's establish the claim. We'll show that the non-empty S_M's ($M \subseteq L$) are disjoint and that the union of the S_M's is $\cap_{i \in \overline{L}} A_i$.

3. Suppose S_M and $S_{M'}$ are non-empty and $M \neq M'$. Let $s \in S_M$. Show $s \notin S_{M'}$.

4. Now we'll show: $\cup_{M \subseteq L} S_M \subseteq \cap_{i \in \overline{L}} A_i$.

   (a) First note that $M \subseteq L$ implies $\bar{L} \subseteq \bar{M}$. Why does $\cap_{i \in \bar{M}} A_i \subseteq \cap_{i \in \bar{L}} A_i$?

   (b) How does this imply the claim?

5. And finally we'll show: $\cap_{i \in \overline{L}} A_i \subseteq \cup_{M \subseteq L} S_M$

   Let $s \in \cap_{i \in \overline{L}} A_i$, $M_1 = \{i \in [n] : s \in A_i\}$ and $M_2 = \{j \in [n] : s \in \bar{A}_j\}$.

   Note that, as defined, M_1 and M_2 are disjoint and $\bar{M}_1 = [n] \setminus M_1 = M_2$. Let $M = M_2 \cap L$.

   (a) Show: $i \in M$ implies $s \in \cap_{i \in M} \bar{A}_i$.

   (b) Show: $i \in \bar{M}$ implies $s \in \cap_{i \in \bar{M}} A_i$.

   (c) Show: $s \in \cup_{M \subseteq L} S_M$.