Inclusion-Exclusion

Let \( S = \{a, b, c, d, e, f\} \). \( A_1 = \{a, b, c, d\} \), \( A_2 = \{b, c, d, e\} \) and \( A_3 = \{a, d, e\} \).

1. Find \( A_1 \cup A_2 \cup A_3 \) and \(|A_1 \cup A_2 \cup A_3|\). We will find them directly first—and then with theory.

   We know \(|A_1 \cup A_2 \cup A_3| = |S| - |A_1 \cup A_2 \cup A_3|\) and \( A_1 \cup A_2 \cup A_3 = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \) (DeMorgan’s Law). So to find \(|A_1 \cup A_2 \cup A_3|\) it is enough to find \(|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|\).

2. Find \( \bar{A}_1 \), \( \bar{A}_2 \) and \( \bar{A}_3 \).

3. Find \( \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \) and \(|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|\).

We will now define an appropriate function \( F \) and calculate \(|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|\) using Mobius Inversion. The secret here, as with the calculation of the Euler Totient Function, is to find a nice partition that allows us to calculate an expression for \( G \).

Let \( S_K = \{s \in S : s \in \cap_{i \in K} A_i \text{ and } s \in \cap_{j \in K} \bar{A}_j\} \), for every \( K \subseteq [3] \), and let \( F(K) = |S_K| \).

We will show that the non-empty \( S_L \)’s \( (L \subseteq K) \) form a partition of \( \cap_{i \in K} A_i \).

We’ll do this for each subset of \([3]\). Note importantly that \( \cap_{i \in \emptyset} A_i = S \).

4. Find \( S_{\{1\}} \) and \( F(\{1\}) \).

5. Find \( S_{\{2\}} \) and \( F(\{2\}) \).

6. Find \( S_{\{3\}} \) and \( F(\{3\}) \).

7. Find \( S_{\{1,2\}} \) and \( F(\{1,2\}) \).
8. Find $S_{\{1,3\}}$ and $F(\{1,3\})$.

9. Find $S_{\{2,3\}}$ and $F(\{2,3\})$.

10. Find $S_\emptyset$ and $F(\emptyset)$.

11. Find $S_{\{1,2,3\}}$ and $F(\{1,2,3\})$.

\[ \text{Let } G(K) = \sum_{L \subseteq K} F(L). \]

12. Find $G(\{1\})$ and check that for $L \subseteq \{1\}$ that non-empty $S_L$’s partition $\cap_{i \in \{1\}} A_i$.

13. Find $G(\{2\})$ and check that for $L \subseteq \{2\}$ that non-empty $S_L$’s partition $\cap_{i \in \{2\}} A_i$.

14. Find $G(\{3\})$ and check that for $L \subseteq \{3\}$ that non-empty $S_L$’s partition $\cap_{i \in \{3\}} A_i$.

15. Find $G(\{1,2\})$ and check that for $L \subseteq \{1,2\}$ that non-empty $S_L$’s partition $\cap_{i \in \{1,2\}} A_i$.

16. Find $G(\{1,3\})$ and check that for $L \subseteq \{1,3\}$ that non-empty $S_L$’s partition $\cap_{i \in \{1,3\}} A_i$.

17. Find $G(\{2,3\})$ and check that for $L \subseteq \{2,3\}$ that non-empty $S_L$’s partition $\cap_{i \in \{2,3\}} A_i$.

18. Find $G(\{1,2,3\})$ and check that for $L \subseteq \{1,2,3\}$ non-empty $S_L$’s partition $\cap_{i \in \{1,2,3\}} A_i$.

19. Find $G(\emptyset)$ and check that for $L \subseteq \emptyset$ that non-empty $S_L$’s partition $\cap_{i \in \emptyset} A_i$. 
We wanted $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$. We see that $F([3]) = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$ directly from the definition. Now we will check the theory, and what we get from that:

20. We know $F([3]) = \sum_{L \subseteq [3]} \mu(L, [3]) G(L)$, and we showed that $G(L) = |\cap_{i \in L} A_i|$. and with a few substitutions we'll get:

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = \sum_{J \subseteq [3]} (-1)^{|J|} |\cap_{j \in J} A_j|.$$ 

Let’s check.

We can now follow these same steps to show that:

$$|S| - |\cup_{i \in [n]} A_i| = |\cap_{i \in [n]} \bar{A}_i| = \sum_{J \subseteq [n]} (-1)^{|J|} |\cap_{j \in J} A_j|.$$