Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 17
Inclusion-Exclusion

Let Let $S = \{a, b, c, d, e, f\}$. $A_1 = \{a, b, c, d\}$, $A_2 = \{b, c, d, e\}$ and $A_3 = \{a, d, e\}$. Let $\bar{A}_i = S \setminus A_i$. The index set is [3]. For any $K \subseteq [3]$ let $\bar{K} = [3] \setminus K$.

1. Find $A_1 \cup A_2 \cup A_3$ and $|A_1 \cup A_2 \cup A_3|$. We will find them directly first—and then with theory.

   We know $|\overline{A_1 \cup A_2 \cup A_3}| = |S| - |A_1 \cup A_2 \cup A_3|$ and $\overline{A_1 \cup A_2 \cup A_3} = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ (DeMorgan's Law). So to find $|A_1 \cup A_2 \cup A_3|$ it is enough to find $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$.

2. Find $\bar{A}_1$, $\bar{A}_2$ and $\bar{A}_3$.

3. Find $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ and $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$.

   We will now define an appropriate function F and calculate $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$ using Mobius Inversion. The secret here, as with the calculation of the Euler Totient Function, is to find a nice partition that allows us to calculate an expression for G.

   Let $S_K = \{s \in S : s \in \cap_{i \in \bar{K}} A_i \text{ and } s \in \cap_{j \in K} \bar{A}_j\}$, for every $K \subseteq [3]$, and let $F(K) = |S_K|$.

   We will show that the non-empty S_L's ($L \subseteq K$) form a partition of $\cap_{i \in \bar{K}} A_i$.

   We'll do this for each subset of [3]. Note importantly that $\cap_{i \in \emptyset} A_i = S$.

4. Find $S_{\{1\}}$ and $F(\{1\})$.

5. Find $S_{\{2\}}$ and $F(\{2\})$.

6. Find $S_{\{3\}}$ and $F(\{3\})$.

7. Find $S_{\{1,2\}}$ and $F(\{1, 2\})$.

8. Find $S_{\{1,3\}}$ and $F(\{1,3\})$.

9. Find $S_{\{2,3\}}$ and $F(\{2,3\})$.

10. Find $S_{\{\emptyset\}}$ and $F(\emptyset)$.

11. Find $S_{\{1,2,3\}}$ and $F(\{1,2,3\})$.

Let $G(K) = \sum_{L \subseteq K} F(L)$.

12. Find $G(\{1\})$ and check that for $L \subseteq \{1\}$ that non-empty S_L's partition $\cap_{i \in \overline{\{1\}}} A_i$.

13. Find $G(\{2\})$ and check that for $L \subseteq \{2\}$ that non-empty S_L's partition $\cap_{i \in \overline{\{2\}}} A_i$.

14. Find $G(\{3\})$ and check that for $L \subseteq \{3\}$ that non-empty S_L's partition $\cap_{i \in \overline{\{3\}}} A_i$.

15. Find $G(\{1,2\})$ and check that for $L \subseteq \{1,2\}$ that non-empty S_L's partition $\cap_{i \in \overline{\{1,2\}}} A_i$.

16. Find $G(\{1,3\})$ and check that for $L \subseteq \{1,3\}$ that non-empty S_L's partition $\cap_{i \in \overline{\{1,3\}}} A_i$.

17. Find $G(\{2,3\})$ and check that for $L \subseteq \{2,3\}$ that non-empty S_L's partition $\cap_{i \in \overline{\{2,3\}}} A_i$.

18. Find $G(\{1,2,3\})$ and check that for $L \subseteq \{1,2,3\}$ non-empty S_L's partition $\cap_{i \in \overline{\{1,2,3\}}} A_i$.

19. Find $G(\emptyset)$ and check that for $L \subseteq \emptyset$ that non-empty S_L's partition $\cap_{i \in \overline{\emptyset}} A_i$.

We wanted $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$. We see that $F([3]) = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$ directly from the definition. Now we will check the theory, and what we get from that:

20. We know $F([3]) = \sum_{L \subseteq [3]} \mu(L, [3]) G(L)$, and we showed that $G(L) = |\cap_{i \in \overline{L}} A_i|$. and with a few substitutions we'll get:

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = \sum_{J \subseteq [3]} (-1)^{|J|} |\cap_{j \in J} A_j|.$$

Let's check.

We can now follow these same steps to show that:

$$|S| - |\cup_{i \in [n]} A_i| = |\cap_{i \in [n]} \bar{A}_i| = \sum_{J \subseteq [n]} (-1)^{|J|} |\cap_{j \in J} A_j|.$$