Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 15
Division Poset

Let $\mathbb{P} = (X, \leq)$ be a poset, $F : X \to \mathbb{R}$, and define $G : X \to \mathbb{R}$ as follows: $G(x) = \sum_{\{y:y\leq x\}} F(y)$.

We showed that, for any poset the mobius function μ is given by:

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \not\leq y \\ -\sum_{x\leq z<y} \mu(x,z) & \text{else} \end{cases}$$

We also showed that $\sum_{x\leq z\leq y} \mu(x,z) = \delta(x,y)$.

Mobius Inversion Theorem:

$$F(x) = \sum_{\{y:y\leq x\}} \mu(y,x) \cdot G(y).$$

Division Poset and Euler's Totient Function

Let $[n] = \{2, 3, \ldots n\}$ and define the divisibility relation "$|$": For $x, y \in X$, $x|y$ (that is, x *divides* y, or y *is divisible by* x) if there is an integer k such that $kx = y$. Then $\mathbb{D} = ([n], |)$ is a poset.

For any positive integer n let $\phi(n)$ be the number of positive integers relatively prime to n.

Let $\mathbb{D}(n) = \{d : d|n\}$. This is a subset of $[n]$, so we can view it as an induced poset of $\mathbb{D} = ([n], |)$ with respect to the divisibility relation. We argued that $\mathbb{D}(p^\alpha)$ is a *linear* poset.

Let $G(n) = \sum_{d|n} \phi(d)$. We showed that $G(n) = n$.

We found $\phi(p) = p - 1$ and $\phi(p^\alpha = p^\alpha - p^{\alpha-1} = p^\alpha \cdot (1 - \frac{1}{p})$. We will now find $\phi(n)$ for general n using the Mobius Inversion Theorem.

Product Posets

Let $\mathbb{P}_1 = (X, \leq_1)$ with mobius function μ_1 and $\mathbb{P}_2 = (X, \leq_2)$ with mobius function μ_2 be posets. We can define a poset on $X \times Y = \{(x,y) : x \in X \wedge y \in Y\}$ by defining the relation:

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

As $\mathbb{P}$ is a poset it has a corresponding mobius function (as given above). It can be shown that:

$$\mu((x_1, y_1), (x_2, y_2)) = \mu_1(x_1, x_2) \cdot \mu(y_1, y_2)).$$

We will produce nice formula for $\phi(n)$ where $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$.

1. Let p and q be primes. Find $\mathbb{D}(p)$, $\mathbb{D}(q)$ and $\mathbb{D}(pq)$.

2. Find $\mathbb{D}(p) \times \mathbb{D}(q)$.

3. Find a bijection from $\mathbb{D}(pq)$ to $\mathbb{D}(p) \times \mathbb{D}(q)$.

   Let μ_p be the mobius function for $\mathbb{D}(p)$. Let μ_q be the mobius function for $\mathbb{D}(q)$. And let μ be the mobius function for $\mathbb{D}(pp)$.

4. Find $\mu(1, pq)$.

5. Find $\mu(p, pq)$.

6. Find $\mu(pq, p)$.

7. Find $\mu(pq, pq)$.