Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 14
Division Poset

Let $\mathbb{P} = (X, \leq)$ be a poset, $F : X \to \mathbb{R}$, and define $G : X \to \mathbb{R}$ as follows: $G(x) = \sum_{\{y:y\leq x\}} F(y)$.

We showed that, for any poset the mobius function μ is given by:

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \not\leq y \\ -\sum_{x\leq z<y} \mu(x,z) & \text{else} \end{cases}$$

We also showed that $\sum_{x\leq z\leq y} \mu(x,z) = \delta(x,y)$.

Mobius Inversion Theorem:

$$F(x) = \sum_{\{y:y\leq x\}} \mu(y,x) \cdot G(y).$$

Division Poset and Euler's Totient Function

Let $[n] = \{2, 3, \ldots n\}$ and define the divisibility relation "$|$": For $x, y \in X$, $x|y$ (that is, *x divides y*, or *y is divisible by x*) if there is an integer k such that $kx = y$. Then $\mathbb{D} = ([n], |)$ is a poset.

For any positive integer n let $\phi(n)$ be the number of positive integers relatively prime to n.

Let $\mathbb{D}(n) = \{d : d|n\}$. This is a subset of $[n]$, so we can view it as an induced poset of $\mathbb{D} = ([n], |)$ with respect to the divisibility relation.

1. Find $\phi(2)$, $\phi(5)$, and $\phi(10)$.

2. Let p be prime. Find $\phi(p)$.

3. Let p be prime. Find $\phi(p^\alpha)$.

4. Argue that $\mathbb{D}(p^\alpha)$ is a *linear* poset.

Now we will find $\phi(n)$ using the Mobius Inversion Theorem. It is overkill if $n = p^\alpha$ but very nice for general n.

Let $G(n) = \sum_{d|n} \phi(d)$.

5. Check that $G(10) = 10$.

Prop. $G(n) = n$.

Proof Steps.

(a) Let $d_1, d_2, \ldots, d_l$ be the divisors of n.

(b) Let $S_{d_i} = \{k \in [n] : \gcd(n, k) = d_i\}$.

(c) Claim: $S_{d_i} \neq \emptyset$.

(d) Claim: $S_{d_1} \cup \ldots \cup S_{d_l} = [n]$.

(e) Claim: $S_{d_i} \cap S_{d_j} = \emptyset$ for $i \neq j$. So $\{S_{d_1}, \ldots, S_{d_l}\}$ is a partition of $[n]$.

(f) Claim: $n = |S_{d_1}| + \ldots + |S_{d_l}|$.

(g) Claim: if $d|n$, $n = dd'$ and $\phi(d) = k$ with $\{r_1, \ldots, r_k\}$ relatively prime to d, then $S_{d'} = \{r_1 d', r_2 d', \ldots, r_k d'\}$.

(h) Claim: if $d|n$ and $n = dd'$then $\phi(d) = |S_{d'}|$.

(i) Claim: $n = \phi(d_1) + \phi(d_2) + \ldots + \phi(d_l)$. $\square$

Product Posets.

Let $\mathbb{P}_1 = (X, \leq_1)$ with mobius function μ_1 and $\mathbb{P}_2 = (X, \leq_2)$ with mobius function μ_2 be posets. We can define a poset on $X \times Y = \{(x, y) : x \in X \wedge y \in Y\}$ by defining the relation:

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

As $\mathbb{P}$ is a poset it has a corresponding mobius function (as given above). It can be shown that:

$$\mu((x_1, y_1), (x_2, y_2)) = \mu_1(x_1, x_2) \cdot \mu(y_1, y_2)).$$

You might guess now how this might apply to finding a nice formula for $\phi(n)$ where $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$.