Last name ______________________

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LARSON—MATH 750—CLASSROOM WORKSHEET 13
Mobius Inversion

For the poset $\mathbb{P} = (X, \leq)$, let $\mathcal{F} = \{f : X \times X \to \mathbb{R}\}$. Then define the *Kronecker delta function*:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and the *zeta function*:

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases}$$

So $\delta, \zeta \in \mathcal{F}$.

For functions $f, g \in \mathcal{F}$ the *convolution product* $f * g \in \mathcal{F}$ is:

$$\begin{cases} 0 & \text{if } x \not\leq y \\ \sum_{\{z: x \leq z \leq y\}} f(x,z)g(z,y) & \text{else} \end{cases}$$

Now let $\mathcal{F}' = \{f \in \mathcal{F} : \forall y \in X \; f(y,y) \neq 0\}$. This will be the *group of invertible functions* on $\mathbb{P}$.

For $f \in \mathcal{F}'$ define g as follows:

$$g(x,y) = \begin{cases} 0 & \text{if } x \not\leq y \\ \frac{1}{f(y,y)} & \text{if } x = y \\ -\frac{1}{f(y,y)} \sum_{\{z: x \leq z < y\}} g(x,z)f(z,y) & \text{else} \end{cases}$$

We'll show: for every $f \in \mathcal{F}'$, g is the inverse (with respect to the convolution product) of f, that is $f * g = \delta$ and $g * f = \delta$.

Since $\zeta \in \mathcal{F}'$ and thus ζ has an inverse, we *define* the *mobius function* μ to be this inverse; that is, μ is the function such that $\zeta * \mu = \mu * \zeta = \delta$.

Let $F : X \to \mathbb{R}$, and define $G : X \to \mathbb{R}$ as follows: $G(x) = \sum_{\{y: y \leq x\}} F(y)$.

Mobius Inversion Theorem:

$$F(x) = \sum_{\{y: y \leq x\}} \mu(y,x) \cdot G(y).$$

We showed that, for any poset the mobius function μ is given by:

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \\ -\sum_{x \leq z < y} \mu(x, z) & \text{else} \end{cases}$$

1. Show that $\sum_{x \leq z \leq y} \mu(x, z) = \delta(x, y)$.

For any positive integer $n > 1$ let $\phi(n)$ be the number of positive integers relatively prime to n.

2. Find $\phi(10)$.

3. Check that: $\phi(10) = 10 \cdot \prod_{p|10}(1 - \frac{1}{p})$