

Last name _____

First name _____

LARSON—MATH 750—CLASSROOM WORKSHEET 11
Convolution, Zeta and Mobius

For the poset $\mathbb{P} = (X, \leq)$, let $\mathcal{F} = \{f : X \times X \rightarrow \mathbb{R}\}$. Then define the *Kronecker delta function*:

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and the *zeta function*:

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases}$$

So $\delta, \zeta \in \mathcal{F}$.

For functions $f, g \in \mathcal{F}$ the *convolution product* $f * g \in \mathcal{F}$ is:

$$\begin{cases} 0 & \text{if } x \not\leq y \\ \sum_{\{z: x \leq z \leq y\}} f(x, z)g(z, y) & \text{else} \end{cases}$$

Now let $\mathcal{F}' = \{f \in \mathcal{F} : \forall y \in X f(y, y) \neq 0\}$. This will be the *group of invertible functions* on \mathbb{P} .

For $f \in \mathcal{F}'$ define g as follows:

$$g(x, y) = \begin{cases} 0 & \text{if } x \not\leq y \\ \frac{1}{f(y, y)} & \text{if } x = y \\ -\frac{1}{f(y, y)} \sum_{\{z: x \leq z < y\}} g(x, z)f(z, y) & \text{else} \end{cases}$$

We'll show: for every $f \in \mathcal{F}'$, g is the inverse (with respect to the convolution product) of f , that is $f * g = \delta$ and $g * f = \delta$.

Since $\zeta \in \mathcal{F}'$ and thus ζ has an inverse, we *define* the *mobius function* μ to be this inverse; that is, μ is the function such that $\zeta * \mu = \mu * \zeta = \delta$.

Let $\mathbb{P} = (\mathcal{P}([3]), \subseteq)$ and let $f = \zeta = \zeta_{\mathbb{P}}$.

1. Find $g(\{1\}, \{1, 2\})$ directly from the definition.

2. Find $g(\{1\}, \{1, 2, 3\})$ directly from the definition.

Let $\mu(A, B) = (-1)^{|B|-|A|}$.

3. Check that $\mu(\{1\}, \{1, 2\})$ is the same as what you calculated for $g(\{1\}, \{1, 2\})$.

4. Check that $\mu(\{1\}, \{1, 2, 3\})$ is the same as what you calculated for $g(\{1\}, \{1, 2, 3\})$.

5. Now make a table with the sets in $\mathcal{P}([3])$ as the indices for the rows x and columns y with table entries, $\mu(x, y) = g(x, y)$.