Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 09
Functions on Posets

Given a poset $\mathbb{P} = (X, \leq)$, we can define the *comparability graph* $G_{\mathbb{P}}$ for $\mathbb{P}$ on X vertex set X and edge relation defined by:

$$x \sim y \text{ in } G_{\mathbb{P}} \text{ if and only if } x \text{ and } y \text{ are comparable in } \mathbb{P}$$

A *clique cover* of a graph G is a partition of the vertex set $V(G)$ into sets which each induce a clique in G. The *clique covering number* $\bar{\chi}$ of a graph is the cardinality of a minimum clique cover of the graph.

Theorem (Mirsky's Theorem). The height of a poset equals the minimum number of anti-chains in a partition of the poset into anti-chains.

Theorem (Dilworth's Theorem). The width of a poset equals the minimum number of chains in a partition of the poset into anti-chains.

A graph is *perfect* if for every subgraph $\alpha = \bar{\chi}$ (or, equivalently, $\omega = \chi$).

An *odd hole* is a graph is an odd cycle with more than 3 vertices. An *odd anti-hole* in a graph is a subgraph whose complement is an odd hole.

The **Weak Perfect Graph Theorem**. A graph is perfect if and only if its complement is perfect.

The **Strong Perfect Graph Theorem**. A graph is perfect if and only if its contains no odd holes nor odd anti-holes.

Proposition. Comparability graphs of posets are perfect.

Delta, Zeta and Mobius Functions

For the poset $\mathbb{P} = (X, \leq)$, let $\mathcal{F} = \{f : X \times X \to \mathbb{R}\}$. Then define the *Kronecker delta function*:

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and the *zeta function*:

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases}$$

So $\delta, \zeta \in \mathcal{F}$.

1. Let $\mathbb{P} = (\mathcal{P}([3]), \subseteq)$. Make a table with the sets in $\mathcal{P}([3])$ as the indices for the rows x and columns y with table entries $\zeta(x, y)$.

2. Let $\mathbb{D} = ([10], |)$. Make a table with the sets in $[10]$ as the indices for the rows x and columns y with table entries $\zeta(x, y)$.

For functions $f, g \in \mathcal{F}$ the *convolution product* $f * g \in \mathcal{F}$ is:

$$\begin{cases} 0 & \text{if } x \not\leq y \\ \sum_{\{z : x \leq z \leq y\}} f(x, z) g(z, y) & \text{else} \end{cases}$$

3. Let $f \in \mathcal{F}$. Find $f * \delta$.

4. Does $f * \delta = \delta * f$?

5. $\mathbb{P} = (\mathcal{P}([3]), \subseteq)$ and $f(A, B) = |B| - |A|$ if $A \subseteq B$ and $f(A, B) = 0$ if $f \not\subseteq B$. Find $(f * \zeta)(\{1\}, [3])$.

6. Find Find $(\zeta * f)(\{1\}, [3])$