Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 08
Comparability Graphs

Given a poset $\mathbb{P} = (X, \leq)$, we can define the *comparability graph* $G_{\mathbb{P}}$ for $\mathbb{P}$ on X vertex set X and edge relation defined by:

$$x \sim y \text{ in } G_{\mathbb{P}} \text{ if and only if } x \text{ and } y \text{ are comparable in } \mathbb{P}$$

Let $[n] = \{1, 2, \ldots, n\}$ with the divisibility relation "$|$": For $m_1, m_2 \in [n]$, $m_1|m_2$ (that is, m_1 *divides* m_2, or m_2 *is divisible by* m_1) if there is an integer k such that $km_1 = m_2$.

1. Draw the comparability graph $G_{\mathbb{P}}$ for the divisibility poset $\mathbb{P} = ([10], |)$.

   A *clique cover* of a graph G is a partition of the vertex set $V(G)$ into sets which each induce a clique in G. The *clique covering number* $\bar{\chi}$ of a graph is the cardinality of a minimum clique cover of the graph.

2. Find the independence number α and clique covering number $\bar{\chi}$ of $G_{\mathbb{P}}$.

3. Find a subgraph of $G_{\mathbb{P}}$ where $\alpha \neq \bar{\chi}$.

Theorem (Mirsky's Theorem). The height of a poset equals the minimum number of anti-chains in a partition of the poset into anti-chains.

Theorem (Dilworth's Theorem). The width of a poset equals the minimum number of chains in a partition of the poset into anti-chains.

A graph is *perfect* if for every subgraph $\alpha = \bar{\chi}$ (or, equivalently, $\omega = \chi$).

An *odd hole* is a graph is an odd cycle with more than 3 vertices. An *odd anti-hole* in a graph is a subgraph whose complement is an odd hole.

The **Weak Perfect Graph Theorem**. A graph is perfect if and only if its complement is perfect.

The **Strong Perfect Graph Theorem**. A graph is perfect if and only if its contains no odd holes nor odd anti-holes.

4. Explain why the anti-chains in a poset $\mathbb{P}$ and the independent sets in $G_{\mathbb{P}}$ are exactly the same.

5. Explain why the chains in a poset $\mathbb{P}$ and the cliques in $G_{\mathbb{P}}$ are exactly the same.

6. Explain why comparability graphs are perfect.