A *chain* in \((X, \leq)\) is a linearly ordered subset of \(X\) (with respect to the given order \(\leq\)), that is \(C \subseteq X\) and \((C, \leq)\) is a linear order. The *height* of a poset is the number of elements in a longest (largest cardinality) chain.

An *anti-chain* in \((X, \leq)\) is a collection of elements which are pair-wise incomparable. (It is easy to find a partition of a finite poset into anti-chains: the set of minimal elements is an anti-chain, so repeatedly take collections of minimal elements). The *width* of a poset is the number of elements of a largest cardinality anti-chain.

**Theorem** (Sperner’s Theorem) Let \(S\) be a set with \(|S| = n\). Let \(\mathcal{F}\) be a family of subsets of \(S\) (so \(\mathcal{F} \subseteq \mathcal{P}(S)\)). Then no anti-chain in the poset \((\mathcal{F}, \subseteq)\) has more than \(\binom{n}{\lfloor \frac{n}{2} \rfloor}\) elements.

**Theorem** (Mirsky’s Theorem). The height of a poset equals the minimum number of anti-chains in a partition of the poset into anti-chains.

**Theorem** (Dilworth’s Theorem). The width of a poset equals the minimum number of chains in a partition of the poset into anti-chains.

Let \([n] = \{1, 2, \ldots, n\}\) with the divisibility relation “\(\mid\)”: For \(m_1, m_2 \in [n]\), \(m_1 \mid m_2\) (that is, \(m_1\) divides \(m_2\), or \(m_2\) is divisible by \(m_1\)) if there is an integer \(k\) such that \(km_1 = m_2\).

1. Find a maximum chain in \(([100], \mid)\).

2. Find a partition of \(([100], \mid)\) into a minimum number of anti-chains.

3. Find a maximum anti-chain in \(([100], \mid)\).

4. Find a partition of \(([100], \mid)\) into a minimum number of chains.
Bad Proof of Dilworth’s Theorem

5. Find the error.

(a) Induction Basis: the Theorem is true for posets with ground set $X$ where $|X| = 1$.

(b) Induction Hypothesis: assume the Theorem is true for posets with ground sets $X$ where $|X| < k$.

(c) Let $P = (X, \leq)$, where $|X| = k$. Let $A$ be a maximum anti-chain. So any chain partition of $P$ must have at least $|A|$ elements (chains).

(d) If $A \neq X$, there is an $x \in X \setminus A$ which is either a maximal or minimal element of $P$. We can assume $x$ is a maximal element.

(e) Let $P' = (X \setminus \{x\}, \leq)$ be the poset induced on the elements of $X \setminus \{x\}$. Note that $|X \setminus \{x\}| = k - 1$.

(f) $A \subseteq X \setminus \{x\}$ is a maximum anti-chain in $P'$.

(g) Applying the inductive hypothesis, we know that $P'$ has a partition $C$ into $|A|$ chains. Let $C = \{C_1, C_2, \ldots, C_{|A|}\}$.

(h) There must be a $y \in X \setminus \{x\}$ such that $x$ covers $y$ in $P$ (otherwise $x$ isn’t comparable to any other element and would have been in $A$).

(i) $y$ is in one of the chains in $C$. Call it $C_y$. Let $C'_y = C_y \cup \{x\}$.

(j) Then $C' = C \setminus \{C_y\} \cup \{C'_y\}$ is a chain partition of $P$ with $|A|$ elements (the minimum possible number). $\Box$