Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 07
Chains & Antichains

A *chain* in $(X, \leq)$ is a linearly ordered subset of X (with respect to the given order $\leq$), that is $C \subseteq X$ and $(C, \leq)$ is a linear order. The *height* of a poset is the number of elements in a longest (largest cardinality) chain.

An *anti-chain* in $(X, \leq)$ is a collection of elements which are pair-wise incomparable. (It is easy to find a partition of a finite poset into anti-chains: the set of minimal elements is an anti-chain, so repeatedly take collections of minimal elements). The *width* of a poset is the number of elements of a largest cardinality anti-chain.

Theorem (Sperner's Theorem) Let S be a set with $|S| = n$. Let $\mathcal{F}$ be a family of subsets of S (so $\mathcal{F} \subseteq \mathcal{P}(S)$). Then no anti-chain in the poset $(\mathcal{F}, \subseteq)$ has more than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements.

Theorem (Mirsky's Theorem). The height of a poset equals the minimum number of anti-chains in a partition of the poset into anti-chains.

Theorem (Dilworth's Theorem). The width of a poset equals the minimum number of chains in a partition of the poset into anti-chains.

Let $[n] = \{1, 2, \ldots, n\}$ with the divisibility relation "$|$": For $m_1, m_2 \in [n]$, $m_1 | m_2$ (that is, m_1 *divides* m_2, or m_2 *is divisible by* m_1) if there is an integer k such that $km_1 = m_2$.

1. Find a maximum chain in $([100], |)$.

2. Find a partition of $([100], |)$ into a minimum number of anti-chains.

3. Find a maximum anti-chain in $([100], |)$.

4. Find a partition of $([100], |)$ into a minimum number of chains.

Bad Proof of Dilworth's Theorem

5. Find the error.

   (a) Induction Basis: the Theorem is true for posets with ground set X where $|X| = 1$.

   (b) Induction Hypothesis: assume the Theorem is true for posets with ground sets X where $|X| < k$.

   (c) Let $\mathbb{P} = (X, \leq)$, where $|X| = k$. Let $\mathcal{A}$ be a maximum anti-chain. So any chain partition of $\mathbb{P}$ must have at least $|\mathcal{A}|$ elements (chains).

   (d) If $\mathcal{A} \neq X$, there is an $x \in X \setminus \mathcal{A}$ which is either a maximal or minimal element of $\mathbb{P}$. We can assume x is a maximal element.

   (e) Let $\mathbb{P}' = (X \setminus \{x\}, \leq)$ be the poset induced on the elements of $X \setminus \{x\}$. Note that $|X \setminus \{x\}| = k - 1$.

   (f) $\mathcal{A} \subseteq X \setminus \{x\}$ is a maximum anti-chain in $\mathbb{P}'$.

   (g) Applying the inductive hypothesis, we know that $\mathbb{P}'$ has a partition $\mathcal{C}$ into $|\mathcal{A}|$ chains. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_{|\mathcal{A}|}\}$.

   (h) There must be a $y \in X \setminus \{x\}$ such that x covers y in $\mathbb{P}$ (otherwise x isn't comparable to any other element and would have been in $\mathcal{A}$).

   (i) y is in one of the chains in $\mathcal{C}$. Call it C_y. Let $C'_y = C_y \cup \{x\}$.

   (j) Then $\mathcal{C}' = \mathcal{C} \setminus \{C_y\} \cup \{C'_y\}$ is a chain partition of $\mathbb{P}$ with $|\mathcal{A}|$ elements (the minimum possible number). $\square$