Last name ______________________

First name ______________________

LARSON—MATH 750—CLASSROOM WORKSHEET 05
Terminology

A *partial order* on a set X is a relation "$\leq$" on X that is reflexive, anti-symmetric and transitive. We call $(X, \leq)$ a *partially ordered set* (or *poset*).

Two elements $x, y \in X$ are *comparable* if either $x \leq y$ or $y \leq x$. Otherwise they are *incomparable*.

A poset $(X, \leq)$ is *linearly (or totally) ordered* if, for every $x, y \in X$, either $x \leq y$ or $y \leq x$, that is, if every pair of elements is comparable.

(1^{st} definition) A *linear extension* of a poset $(X, \leq)$ is a linearly ordered poset $(X, \leq')$ with the same elements.

A *minimal element* in a poset $(X, \leq)$ is an element $x \in X$ such that *if* there is an element $z \in X$ with $z \leq x$ then $z = x$ (that is, there is no element comparable to x that is less than x); x is a *minimum element* (or *bottom*) if for every $z \in X$, $x \leq z$. A finite poset has minimal elements; it may not have a minimum element.

Similarly, a *maximal element* in a poset $(X, \leq)$ is an element $x \in X$ such that *if* there is an element $z \in X$ with $x \leq z$ then $z = x$ (that is, there is no element comparable to x that is greater than x); x is a *maximum element* (or *top*) if for every $z \in X$, $z \leq x$. A finite poset has minimal elements; it may not have a minimum element.

A *chain* in $(X, \leq)$ is a linearly ordered subset of X (with respect to the given order $\leq$), that is $C \subseteq X$ and $(C, \leq)$ is a linear order.

An *anti-chain* in $(X, \leq)$ is a collection of elements which are pair-wise incomparable. (It is easy to find a partition of a finite poset into anti-chains: the set of minimal elements is an anti-chain, so repeatedly take collections of minimal elements). The *height* of a poset is the number of elements in a longest chain.

Theorem. There is a linear extension of any (finite) poset.

Theorem (Mirsky's Theorem). The height of a (finite) poset equals the number of anti-chains in a minimum partition of the poset into anti-chains.

1. Let X be the set of non-empty subsets of $[4] = \{1, 2, 3, 4\}$ with the inclusion relation $\subseteq$. Explain why $(X, \subseteq)$ is not a linear order.

2. Draw the Hasse diagram for $(X, \subseteq)$.

3. Find the set of minimal elements for $(X, \subseteq)$. Is there a minimum element?

4. Find the set of maximal elements for $(X, \subseteq)$. Is there a maximum element?

5. Find a longest chain in $(X, \subseteq)$ and its height.

6. Find a maximum anti-chain in $(X, \subseteq)$.

7. Find a partition of $(X, \subseteq)$ into a minimum number of anti-chains.

8. Find a linear extension of $(X, \subseteq)$.