Last name ______________________

First name ______________________

LARSON—OPER 731–SAGE WORKSHEET 06
Polytopes!

1. Log in to your Sage Cloud account.

   (a) Start Chrome browser.

   (b) Go to `http://cocalc.com`

   (c) Click "Sign In".

   (d) Click project **Classroom Worksheets**.

   (e) Click "New", call it **s06**, then click "Sage Worksheet".

   First we'll define a polytope in Sage using inequalities. We know that a polytope is defined as a convex hull of a point set—but that it can also be described in terms of a system of linear inequalities (or *hyperplanes*, as an intersection of *half-spaces*). The Weyl and Minkowski Theorems establish the interchangeability of these descriptions. Sometimes they are distinguished at H-polytopes and V-polytopes depending on whether their descriptions are given as **H**yperplanes or **V**ertices.

   In Sage the inequalities are input in a standardized (independent of variable name choice) but unfamiliar form.

   If the polytope is defined in terms of inequalities with two variables the inequalities will be represented by coefficient lists with three entries. The inequalities are assumed to be of the form of a variable expression, $\geq$, followed by a constant (if yours isn't just fix it up). So $x+y \leq 1$ becomes $-x-y \geq -1$. The inequality $a_1x_1+a_2x_2+\ldots a_nx_n \geq b$ is represented by the list `[-b,`$a_1, a_2 \ldots, a_n$`]`. So, $x+y \leq 1$ (in a system with a total of two variables) is represented as: `[1,-1,-1]`.

2. Consider the system $x+y \leq 1$, $x \geq 0$, $y \geq 0$. The polytope is formed using the `Polyhedron` constructor with the parameter `ieqs` ("inequalities") followed by a list of lists representing each of these three inequalities.

   Evaluate: `P = Polyhedron(ieqs = [[0,0,1],[0,1,0],[1,-1,-1]])`. P is the name of the defined polytope. Evaluate `P` to see what kind of object Sage thinks it is.

3. In small dimensions (≤ 3) we can visualize polytopes (typically called *polygons* in 2-dimensions, *polyhedron* in 3-dimensions, and *polytopes* in the general case). Evaluate `P.show()`.

4. Now that we have this object Sage can tell us many things about it, for instance what its extreme points (or *vertices*) are. Evaluate: `P.vertices()`. What do you get?

5. To find the dimension of this polytope, evaluate `P.dimension()`.

6. So the facets of P will be faces with 1-dimension. The `faces` method can be used to find all the faces (of every dimension). To find the 1-dimensional faces and then list them as inequalities (H-representation), evaluate:

```
facets = P.faces(1)
for f in facets:
    f.ambient_Hrepresentation()
```

Now lets try this familiar polytope $P2$ defined by the following inequalities:

$$\begin{array}{ccccccc} x_1 & + & x_2 & & & \leq & 1 \\ x_1 & + & & & x_3 & \leq & 1 \\ & & x_2 & + & x_3 & \leq & 1 \\ & & & 0 \leq x_i \leq 1 & & & \end{array}$$

7. Evaluate: `P2 = Polyhedron(ieqs=[[1,-1,-1,0],[1,-1,0,-1],[1,0,-1,-1], [0,1,0,0],[0,0,1,0],[0,0,0,1],[1,-1,0,0],[1,0,-1,0],[1,0,0,-1]])`, and `P2.show()`.

8. Find its extreme points with: `P2.vertices()`.

9. Find its dimension with: `P2.dimension()`.

10. Find its facets with:

```
facets = P2.faces(2)
for f in facets:
    f.ambient_Hrepresentation()
```

11. We can also define a polytope using the `Polyhedron()` constructor with a list of finite points. Evaluate: `P3 = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])`. Then try `P3.show()`

12. We can get back a representation of $P3$ by inequalities with: `P3.Hrepresentation()`. Here x is a vector with 2 variables. Can you interpret the output?

13. (**Bonus**). See if you can code some other polytope (maybe the one from the last test) into Sage and see what information you can learn about it.