Organizational Notes

1. Don’t forget to send your Notes / Classroom worksheet after each class (make the email subject useful: like “Math 656 c19 notes”).

2. The VCU Discrete Math Seminar is every Wednesday.

3. h06 is due on Wednesday (#3.3.1, 3.3.2, 3.3.3, 3.3.6, 3.3.10).

4. Read ahead! Next up we’ll talk about Petersen’s Theorem (Corollary to Tutte’s Theorem in Sec 3.3) and then Network Flow problems (Sec. 4.3)

Concepts & Notation

• factor-critical graph, near-perfect matching, Edmonds-Gallai Decomposition (West paper).

• Petersen’s Theorem (Sec. 3.3).

• Network Flows (Sec. 4.3).

Review

1. (Theorem) (Berge-Tutte Formula) \( \nu = \frac{1}{2}(n - df(G)) \).

2. (Theorem) (Tutte’s Theorem) A graph \( G \) has a perfect matching if and only if for every \( S \subseteq V(G) \) \( o(G - S) \leq |S| \).

Notes

1. What is a factor-critical graph?

2. What is a near-perfect matching?

3. A vertex \( v \) in a graph is either (1) covered by every maximum matching (set \( B \)), or (2) not covered by every maximum matching (set \( D \)). A vertex in \( B \) either (1) has a neighbor outside \( B \) (set \( A \)) or (2) does not (set \( C \)). The Gallai-Edmonds Decomposition is the partition of \( V(G) \) into sets \( C, A \) and \( D \).

4. Find the Gallai-Edmonds Decomposition for a graph with a perfect matching.

5. Find the Gallai-Edmonds Decomposition for \( P_3 \).

7. Find the Gallai-Edmonds Decomposition for the house graph.

8. Find the Gallai-Edmonds Decomposition for the graph formed by the join of $3K_3$ and $P_2$.

9. **(Gallai-Edmonds Structure Theorem).** Let $A, C, D,$ be the sets in the Gallai-Edmonds Decomposition of a graph $G$. Let $G_1, \ldots, G_k$ be the components of $G[D]$. If $M$ is a maximum matching in $G$ then:

   (a) $M$ covers $C$ and matches $A$ into distinct components of $G[D]$.
   (b) Each $G_i$ is factor-critical and $M$ restricts to a near-perfect matching on $G_i$.
   (c) If $S \subseteq A$ is non-empty then $N_G(S)$ has a vertex in at least $|S| + 1$ of $G_1, \ldots, G_k$.
   (d) $\text{def}(A) = \text{def}(G) = k - |A|$.

10. What does the Gallai-Edmonds Structure Theorem say for a graph with a perfect matching? Find a maximum matching $M$ and check. Try some non-empty subsets $S \subseteq A$. Find $\text{def}(A)$, $\text{def}(G)$, $k$.

11. What does the Gallai-Edmonds Structure Theorem say for $P_3$? Find a maximum matching $M$ and check. Try some non-empty subsets $S \subseteq A$. Find $\text{def}(A)$, $\text{def}(G)$, $k$.

12. What does the Gallai-Edmonds Structure Theorem say for $S_4$? Find a maximum matching $M$ and check. Try some non-empty subsets $S \subseteq A$. Find $\text{def}(A)$, $\text{def}(G)$, $k$.

13. What does the Gallai-Edmonds Structure Theorem say for the house graph? Find a maximum matching $M$ and check. Try some non-empty subsets $S \subseteq A$. Find $\text{def}(A)$, $\text{def}(G)$, $k$.

14. What does the Gallai-Edmonds Structure Theorem say for the graph formed by the join of $3K_3$ and $P_2$? Find a maximum matching $M$ and check. Try some non-empty subsets $S \subseteq A$. Find $\text{def}(A)$, $\text{def}(G)$, $k$. 