You should know the following definitions, theorems, examples, and proofs for the test. Write out careful definitions, theorem statements, proofs, and solutions.

Definitions & Notation

1. \( \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{C}, \mathbb{F}_2 \).

2. What are the main properties of a field? Explain why \( \mathbb{Z} \) is not a field.

3. algebraically closed field.

4. monomial in \( x_1, \ldots, x_n \).

5. total degree of a monomial.

6. polynomial in \( x_1, \ldots, x_n \).

7. zero polynomial.

8. \( k^n \), for a field \( k \).

9. Explain how any polynomial \( f \) in variables \( x_1, \ldots, x_n \) defines a function from \( k^n \) to \( k \).

10. What is a zero function? What does it mean for a polynomial \( f \) to define a zero function?

11. \( k[x_1, \ldots, x_n] \).

12. What properties does \( k[x_1, \ldots, x_n] \) have? Is it a field?

13. \( k(x_1, \ldots, x_n) \).

14. Affine variety \( \mathbb{V}(f_1, \ldots, f_n) \).

15. linear variety.

16. twisted cubic.

17. ideal in \( k[x_1, \ldots, x_n] \).

18. \( \langle f_1, \ldots, f_s \rangle \) for \( f_1, \ldots, f_s \) in \( k[x_1, \ldots, x_n] \).

Theorems

19. Relationship between zero polynomials and zero functions.

20. The union and intersection of varieties theorem (with the formulas for these).

Examples & Problems

22. Give some examples of both finite and infinite fields.

23. Find a monomial in \( x, y, z \) with total degree 10.

24. Give an example of a non-zero polynomial which defines a zero function.

25. Give an explicit description of \( \mathbb{R}[x, y] \).

26. Show how the polynomial \( x^2y^3z^5 + yz + xz^2 + xy + xyz + y^2z^3 \) in \( \mathbb{R}[x, y, z] \) can be written in the form \( \sum g_i(y, z)x^i \). Find the \( g_i \)'s.

27. Draw \( \mathbb{V}(xy, xz) \). Explain how you got it.

28. Write the defining equations for \( \mathbb{V}(y - x - 3) \).

29. Draw \( \mathbb{V}(y - x - 3) \).

30. Parameterize the points on \( \mathbb{V}(y - x - 3) \).

31. Show that \( x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2} \) gives a parameterization of the unit circle \( x^2 + y^2 = 1 \).

32. Give an example of a finite variety.

33. Give an example of an empty variety.

34. Give an example of a variety with 2 different sets of defining polynomials.

35. Define \( \langle x^2 - 4, y^2 - 1 \rangle \).

36. Define \( \langle x + y, x - y \rangle \) in \( \mathbb{R}[x, y] \).

37. Show \( x^2 \) in \( \langle x + y, x - y \rangle \).

Proofs

38. Show \( \langle x, y \rangle = \langle x + y, x - y \rangle \).

39. Show \( \langle f_1, \ldots, f_s \rangle \) is an ideal in \( k[x_1, \ldots, x_n] \).