Ideals of Varieties

We’ll show that $I(\mathbb{V}(y - x^2, z - x^3)) = \langle y - x^2, z - x^3 \rangle$.

1. What is $\mathbb{V}(y - x^2, z - x^3)$? Describe this object in words.

2. What is $I(\mathbb{V}(y - x^2, z - x^3))$? Describe this object in words.

3. What is $\langle y - x^2, z - x^3 \rangle$? What is this object, according to the definition?

4. Let $f \in \langle y - x^2, z - x^3 \rangle$, and $(a, b, c) \in \mathbb{V}(y - x^2, z - x^3)$. Argue that $f(a, b, c) = 0$.

5. Explain why the previous result shows that $\langle y - x^2, z - x^3 \rangle \subseteq I(\mathbb{V}(y - x^2, z - x^3))$.

Now let $f \in I(\mathbb{V}(y - x^2, z - x^3)) \subseteq \mathbb{R}[x, y, z]$. $f \in \mathbb{R}[x, y, z]$ means that $f = \sum a_{\alpha, \beta, \gamma} x^\alpha y^\beta z^\gamma$, for some coefficients $a_{\alpha, \beta, \gamma} \in \mathbb{R}$ and exponents $\alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}$.

We will show that each monomial $x^\alpha y^\beta z^\gamma$ in $f$ can be written as a polynomial combination of $y - x^2$ and $z - x^3$. 
6. Explain why \( x^\alpha y^\beta z^\gamma = x^\alpha (x^2 + (y - x^2))^\beta (x^3 + (z - x^3))^\gamma \).

The Binomial Theorem says that \((a + b)^k = \sum_{i=0}^{k} (\binom{k}{i}) a^i b^{k-i}\).

7. Use the Binomial Theorem to show that \((x^2 + (y - x^2))^\beta\) equals \(x^{2\beta}\) plus \((y - x^2)\) terms.

8. Use the Binomial Theorem to show that \((x^3 + (z - x^3))^\gamma\) equals \(x^{3\gamma}\) plus \((z - x^3)\) terms.

These last two results show that each monomial of \(f\) (and thus \(f\)) can be written in the form \((y - x^2)f_1 + (z - x^3)f_2 + r(x)\) for some polynomials \(f_1, f_2 \in \mathbb{R}[x, y, z]\) and \(r(x) \in \mathbb{R}[x]\).

Recall: we showed that \(\nabla(y - x^2, z - x^3) = \{(t, t^2, t^3) : r \in \mathbb{R}\}\).

9. Show that \(f(t, t^2, t^3) = 0\) implies that \(r(t) = 0\) for every \(t \in \mathbb{R}\).

10. Why does this imply that \(r(x)\) is the zero polynomial?

So \(f = (y - x^2)f_1 + (z - x^3)f_2\) and, thus, \(f \in \langle y - x^2, z - x^3 \rangle\)!