Last name _____

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LARSON—MATH 591—BONUS WORKSHEET 01
Ideals of Varieties

We'll show that $I(\mathbb{V}(y - x^2, z - x^3)) = \langle y - x^2, z - x^3 \rangle$.

1. What is $\mathbb{V}(y - x^2, z - x^3)$? Describe this object in words.

2. What is $I(\mathbb{V}(y - x^2, z - x^3))$? Describe this object in words.

3. What is $\langle y - x^2, z - x^3 \rangle$? What is this object, according to the definition?

4. Let $f \in \langle y - x^2, z - x^3 \rangle$, and $(a, b, c) \in \mathbb{V}(y - x^2, z - x^3)$. Argue that $f(a, b, c) = 0$.

5. Explain why the previous result shows that $\langle y - x^2, z - x^3 \rangle \subseteq I(\mathbb{V}(y - x^2, z - x^3))$.

Now let $f \in I(\mathbb{V}(y - x^2, z - x^3)) \subseteq \mathbb{R}[x, y, z]$. $f \in \mathbb{R}[x, y, z]$ *means* that $f = \sum a_{\alpha,\beta,\gamma} x^\alpha y^\beta z^\gamma$, for some coefficients $a_{\alpha,\beta,\gamma} \in \mathbb{R}$ and exponents $\alpha, \beta, \gamma \in \mathbb{Z}^{\geq 0}$.

We will show that each monomial $x^\alpha y^\beta z^\gamma$ in f can be written as a polynomial combination of $y - x^2$ and $z - x^3$.

6. Explain why $x^\alpha y^\beta z^\gamma = x^\alpha(x^2 + (y - x^2))^\beta(x^3 + (z - x^3))^\gamma$.

The *Binomial Theorem* says that $(a + b)^k = \sum_{i=0}^{i=k} \binom{k}{i} a^i b^{k-i}$.

7. Use the Binomial Theorem to show that $(x^2 + (y - x^2))^\beta$ equals $x^{2\beta}$ plus $(y - x^2)$ terms.

8. Use the Binomial Theorem to show that $(x^3 + (z - x^3))^\gamma$ equals $x^{3\gamma}$ plus $(z - x^3)$ terms.

These last two results show that each monomial of f (and thus f) can be written in the form $(y - x^2)f_1 + (z - x^3)f_2 + r(x)$ for some polynomials $f_1, f_2 \in \mathbb{R}[x, y, z]$ and $r(x) \in \mathbb{R}[x]$.

Recall: we showed that $\mathbb{V}(y - x^2, z - x^3) = \{(t, t^2, t^3) : r \in \mathbb{R}\}$.

9. Show that $f(t, t^2, t^3) = 0$ implies that $r(t) = 0$ for every $t \in \mathbb{R}$.

10. Why does this imply that $r(x)$ is the zero polynomial?

So $f = (y - x^2)f_1 + (z - x^3)f_2$ and, thus, $f \in \langle y - x^2, z - x^3 \rangle$!