You should know the following definitions, theorems, algorithms, and proofs for the test. Write out careful definitions, theorem statements, algorithms, proofs, and solutions. Turn these in at test time.

1. Give a definition for each of the following concepts and an example illustrating the concept.

**Definitions & Examples**

Matching, matched vertices, saturates, \( M \)-saturated, perfect matching, maximum matching, \( M \)-alternating path, \( M \)-augmenting path, 1-factor, 2-factor, neighbors, covering, k-edge-coloring, proper k-edge-coloring, k-edge-colorable, edge chromatic number, value of a coloring, optimal \( k \)-coloring, covering, covering number, independent set, independence number, edge independence number, edge covering, edge covering number, \( k \)-vertex coloring, proper (vertex) coloring, chromatic number, clique, clique number.

**Theorems**

Carefully state the following theorems.

(a) Berge’s Theorem
(b) Hall’s Theorem
(c) Marriage Theorem
(d) König’s Theorem
(e) Tutte’s Theorem
(f) Petersen’s Theorem
(g) König’s Edge-Coloring Theorem
(h) Vizing’s Theorem
(i) Gallai Identites
(j) Brooks’ Theorem

**Notation**

2. Give a definition for each of the following notations and an example illustrating its use.

\[ N_G(S), \chi, \chi', \beta, \beta', \alpha, \alpha' \]

**Algorithms**

3. \( \Delta \)-edge-color a bipartite graph.

4. Tarjan-Trojanowski algorithm.

**Proofs**

5. Prove Berge’s Theorem.

6. Prove the Marriage Theorem (assuming Hall’s Theorem).
Problems

7. Show every $k$-cube has a perfect matching.

8. Show that a tree has at most one perfect matching.

9. Find an example of a 5-regular graph that has no perfect matching.

10. Show that Petersen’s Theorem requires the cut-edge condition (find an example with a cut-edge where the theorem fails).

11. Show that a tree has a perfect matching iff $o(G - v) = 1 \forall v \in V(G)$.

12. Show that, for a nonempty regular graph $G$ with odd order, $\chi'(G) = \Delta + 1$.

13. Show that $G$ is bipartite iff $\alpha(H) \geq \frac{1}{2} \nu(H)$ for every subgraph $H$ of $G$.

14. Find a maximum matching $M$ in the above graph and argue that the matching you found is maximum. Find $\alpha'$.

15. Find a minimum cover $C$ in this graph. Find $\beta$.

16. Use Tutte’s Theorem to show that the complete bipartite graph $K_{2,4}$ does not have a perfect matching.

17. Show that the chromatic number of the Petersen graph is 3.

18. Find the clique number of any (nontrivial) bipartite graph $G$. Explain.

19. Find the chromatic number of any (nontrivial) bipartite graph $G$. Explain.

20. A graph is perfect if the clique number and chromatic number of every subgraph are equal. Explain why bipartite graphs are perfect.

21. Find an example of a perfect non-bipartite graph.