Last name _____

First name _____

LARSON—MATH 525—CLASSROOM WORKSHEET 29

Symmetries and Colorings

1. Label the corners of the triangle 1, 2 and 3 and list these 8 colorings as 3-tuples with the color on corner i in the i^{th} place. So (R, R, B), etc. Let $\mathcal{C}$ be the collection of colorings.

2. Let $X = [3]$ be the set of corners of an equilateral triangle. Let G be the symmetry group for X (which we showed was $G = D_3 = S_3$). Let $\mathcal{C}$ be the collection of colorings of X using colors R and B. Each coloring $c \in \mathcal{C}$ can be viewed as a function $c : X \to \mathcal{C}$. So $c(i)$ is the color of the i^{th} corner of X.

   For every permutation $f \in G$ and coloring $c \in \mathcal{C}$ we can define a new coloring function $f * c$ as follows;

   $$f * c : [n] \to \mathcal{C}$$

   with $(f * c)(k) = c(f(k))$, that is, if $f(k)$ is colored "R" in the original coloring c, then k is colored "R" in the new coloring $f * c$.

   Let's see how this works. Let c be the coloring (R, R, B). Or, more exactly, c is a function which could be represented as

   $$c = \begin{pmatrix} 1 & 2 & 3 \\ R & R & B \end{pmatrix}.$$

   Let f be the following permutation from the symmetry group of X:

   $$f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

   $f * c$ will be a new coloring of $X = [3]$. So we need to fill in:

   $$f * c = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}.$$

   Then, $f * c$ is defined as:

   $$f * c = \begin{pmatrix} 1 & 2 & 3 \\ c(f(1)) & c(f(2)) & c(f(3)) \end{pmatrix}.$$

So now we can go back and fill in $f * c$:

$$f * c = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}.$$

What is the relationship between the colorings c and $f * c$? That is, how can you explain things geometrically?