Last name ______________________

First name ______________________

LARSON—MATH 401—Homework 07.

Let G be a commutative group and H be a subgroup of G. Let $G/H = \{g + H : g \in G\}$ be the set of cosets of H in G.

We defined an "addition" for G/H as follows:

$$(g_1 + H) + (g_2 + H) = (g_1 + g_2) + H.$$

We showed that with this addition $(G/H, +)$ is associative, has an additive identity and has additive inverses. So the 3 group axioms are satisfied.

Here's the detail we missed: we still have to show that this addition is well-defined: these cosets are infinite. We defined the addition with respect to particular elements of those infinite collections. We need to show that the choice of representative is trrelevant.

1. How is the coset $g_1 + H$ defined?

2. How is the coset $(g_1 + g_2) + H$ defined?

3. Let $g \in g_1 + H$ and $g' \in g_2 + H$. So $g = g_1 + h_1$ for some $h_1 \in H$ and $g' = g_2 + h_2$ for some $h_2 \in H$. What does $g + g'$ equal?

4. Argue that $g + g' = (g_1 + g_2) + (h_1 + h_2)$.

5. What special property of the group G was necessary for your explanation?

6. Explain why $g + g' \in (g_1 + g_2) + H$.

Let $(G_1, +_1)$ and $(G_2, +_2)$ be groups.

G_1 is *isomorphic* to G_2 is there is a bijection which preserves the group operation. Formally, this means there is a bijection $\phi : G_1 \to G_2$ such that:

$$\text{For any } g, g' \in G_1 \;\; \phi(g +_1 g') = \phi(g) +_2 \phi(g').$$

We will now show that $(\mathbb{Z}, +)$ and $(n\mathbb{Z}, +)$ are isomorphic (for any $n \in \mathbb{N}$).

First define an appropriate bijection $\phi : \mathbb{Z} \to n\mathbb{Z}$, defined by $\phi(k) = nk$, for any $k \in \mathbb{Z}$.

7. Show that ϕ is a bijection.

8. Show that ϕ preserves the group operation (the "addition" in each structure).