Finite Fields.

$F$ is a field (any field). $F[x]$ is the ring of polynomials over $F$.

Let $f(x) \in F[x]$. $\langle f(x) \rangle$ is a maximal ideal if and only if $f(x)$ is irreducible.

$F[x]/I$ is a field iff $I$ is a maximal ideal iff $I = \langle f(x) \rangle$ and $f(x)$ is irreducible.

If $F$ is a finite field and $f(x)$ is an irreducible polynomial over $F$ then $F[x]/\langle f(x) \rangle$ is a finite field.

We showed that $x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$.

1. So what can you conclude about $\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$?

2. What is $\langle x^2 + 1 \rangle$ (that is, what does this mean formally—according to the definition)?

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4. What can you say about the equivalence classes in $\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$, that is, what can you say about them more intuitively (akin to our description of the equivalence classes in $\mathbb{Z}_n/\langle n \rangle = \mathbb{Z}_n/n\mathbb{Z}$)?

5. So what are the elements of $\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$?
6. Make an addition table for \( \mathbb{Z}_3[x]/\langle x^2 + 1 \rangle \).

7. Check that \( (\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle, +) \) is an abelian group.

8. Make a multiplication table for \( \mathbb{Z}_3[x]/\langle x^2 + 1 \rangle \).

9. Check that \( (\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle, \cdot) \) is an abelian group with respect to multiplication.

10. We know theoretically that \( \mathbb{Z}_3[x]/\langle x^2 + 1 \rangle \) is a field. In checking the details what detail remains to check? Why don’t we have to check?