Last name ______________________

First name ______________________

LARSON—MATH 401—CLASSROOM WORKSHEET 34
Polynomial Rings.

Let R be a commutative ring with identity. Define $R[x]$ to be the set of *polynomials* with *indeterminate* x and *coefficients* in R. That is,

$$R[x] = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0 : a_i \in R\}.$$

We say any element in $R[x]$ is a *polynomial over* R. The *degree* of a polynomial is the exponent of the largest-exponent term with a non-zero coefficient (called the *leading term*). A polynomial is *monic* is the coefficient of the leading term is 1.

Addition of polynomials is defined coordinate-wise (adding coefficients of terms of the same degree). We will have to show that $R[x]$ is a commutative ring with identity with respect to this addition.

We define multiplication as follows: for $a(x), b(x) \in R[x]$, with degrees n, m, respectively, $a(x) \cdot b(x) = \sum_{i=0}^{i=n+m} c_i x^i$, where $c_i = \sum_{i=j+k} a_j b_k$ (that is, all possible products of coefficients of pairs $a_j x^j$ and $b_k x^k$ where the exponents sum to i).

Let $a(x) = 5x^3 + 6x^2 - 3x + 4$, and $b(x) = x - 2$ be polynomials over $\mathbb{Z}_7$.

The Division Algorithm for polynomials over a field that says, for any polynomial $a(x)$ and non-constant polynomial $b(x)$, there are unique polynomials $q(x), r(x)$ with $a(x) = q(x) \cdot b(x) + r(x)$, with either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $b(x)$.

1. Divide $a(x)$ by $b(x)$ in the ordinary way to find the quotient $q(x)$ and remainder $r(x)$ (still over $\mathbb{Z}_7$). And check.

A polynomial $p(x)$ is *irreducible* in $R[x]$ if there are not non-constant polynomials $a(x), b(x) \in R[x]$ with $p(x) = a(x) \cdot b(x)$.

2. Show that $p(x) = x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$.

An element $a \in R$ is a *zero* (or *root*) of a polynomial $p(x)$ over R if $p(a) = 0$.

3. Let $p(x) = 2x + 2$ be a polynomial in in $\mathbb{Z}_3[x]$. Find all zeros of $p(x)$ in $\mathbb{Z}_3$.

The **Factors-Zeros Theorem** says: If F is a field and $a \in F$ then $x - a$ is a factor of $p(x)$ if and only if a is a zero of $p(x)$.

4. Use the Factors-Zeros Theorem to show that $p(x) = x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$.