Last name ______________________

First name ______________________

LARSON—MATH 401—CLASSROOM WORKSHEET 29
Ideals & Ring Homomorhisms.

A *ring* R is a set with 2 closed binary relations satisfying certain properties. The operations are usually written "+" and "$\cdot$", as a ring is a generalization of the integers $(\mathbb{Z}, +, \cdot)$:

1. $(R, +)$ is an abelian group: so there is an additive identity 0, and additive inverses.
2. $(R, \cdot)$ is associative.
3. Distributive properties: for every $a, b, c \in R$ $(a + b) \cdot c = a \cdot c + b \cdot c$, and $a \cdot (b + c) = a \cdot b + a \cdot c$.

Other properties that a ring R might have include:

1. **commutative** - if the multiplication is commutative.
2. **with identity** - if there is a multiplicative identity (or "1").
3. **no zero divisors** - if $ab = 0$ implies $a = 0$ or $b = 0$.
4. **integral domain** - if commutative with identity and no zero divisors.
5. **division ring** - if every non-zero element has a multiplicative inverse (and hence (with multiplicative identity).
6. **field** - if a commutative division ring (with multiplicative identity).

We proved:

An *ideal* of a ring R is a subset I that is closed with respect to addition and, for every $x \in I$ and $r \in R$, both $rx, xr \in I$ (closed with respect to left and right multiplication by elements of R).

Let $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ be rings. A *(ring) homomorphism* from R_1 to R_2 is a function which preserves the ring operations. Formally, this means there is a function $\phi : R_1 \to R_2$ such that:

$$\text{For any } r, r' \in R_1 \;\; \phi(r +_1 r') = \phi(r) +_2 \phi(r'), \text{and}$$

$$\text{For any } r, r' \in R_1 \;\; \phi(r \cdot_1 r') = \phi(r) \cdot_2 \phi(r').$$

An ideal I of a ring R is *proper* if it does not equal R.

An ideal I is maximal if the only ideal it is a subset of with more elements is R.

We showed that $5\mathbb{Z}$ is an ideal of $\mathbb{Z}$. We will show that $5\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$.

1. Let I be an ideal of $\mathbb{Z}$ containing every element of $5\mathbb{Z}$ and at least one other element a. Argue that I must contain 1.

2. Suppose an ideal I contains 1. Argue that it is R.