Let \((G_1, +_1)\) and \((G_2, +_2)\) be groups.

A homomorphism from \(G_1\) to \(G_2\) if there is a function which preserves the group operation. Formally, this means there is a function \(\phi : G_1 \rightarrow G_2\) such that:

\[
\text{For any } g, g' \in G_1, \quad \phi(g +_1 g') = \phi(g) +_2 \phi(g').
\]

Let \(e_1\) be the identity element in \(G_1\) and \(e_2\) be the identity element in \(G_2\). We showed that for any homomorphism \(\phi : G_1 \rightarrow G_2\), we have \(\phi(e_1) = e_2\).

Let the kernel of \(\phi\) (\(\ker\phi\)) be the set of elements that map to \(e_2\). Formally,

\[
\ker\phi = \{g \in G : \phi(g) = e_2\}.
\]

So, we showed \(e_1 \in \ker\phi\). We will show that \(\ker\phi\) is a subgroup of \(G\).

Let \(\phi(G)\) be the image of \(\phi\), the set of elements of \(H\) that elements of \(G\) are mapped to. Formally,

\[
\phi(G) = \{\phi(g) : g \in G\}.
\]

It’s true, but we’ll leave it as an exercise to show, that \(\phi(G)\) is a group (and so a subgroup of \(H\)).

1. Let \(G\) and \(H\) be arbitrary groups with \(\phi\) defined so that every element of \(G\) is mapped to \(e_2\). That is, for every \(g \in G\), \(\phi(g) = e_2\). Show \(\phi\) is a homomorphism.

2. Find \(\ker\phi\).
3. Let $G$ be an arbitrary group. Let $\phi$ be the identity isomorphism that maps every element of $G$ to itself. So, $\phi : G \to G$ and, for every $g \in G$, $\phi(g) = g$. Find $\ker \phi$.

1st Isomorphism Theorem. If $G$ is an abelian group, $H$ is any group, and $\phi : G \to H$ is a homomorphism, then $G/\ker \phi \cong \phi(G)$. (If $\phi$ is onto—so $\phi(G) = H$—then $G/\ker \phi \cong H$).

4. Here’s the simplest and still useful application. Let $G$ be an arbitrary group. Let $\phi$ be the identity isomorphism that maps every element of $G$ to itself. So, $\phi : G \to G$ and, for every $g \in G$, $\phi(g) = g$. What does the 1st Isomorphism Theorem say in this case?