Last name ______________________

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LARSON—MATH 401—CLASSROOM WORKSHEET 23
Group Homomorphisms.

Let $(G_1, +_1)$ and $(G_2, +_2)$ be groups.

A *homomorphism* from G_1 to G_2 if there is a function which preserves the group operation. Formally, this means there is a function $\phi : G_1 \to G_2$ such that:

$$\text{For any } g, g' \in G_1 \;\; \phi(g +_1 g') = \phi(g) +_2 \phi(g').$$

Let e_1 be the identity element in G_1 and e_2 be the identity element in G_2. We showed that for *any* homomorphism $\phi : G_1 \to G_2$, we have $\phi(e_1) = e_2$.

Let the *kernel* of ϕ ($\ker \phi$) be the set of elements that map to e_2. Formally,

$$\ker \phi = \{g \in G : \phi(g) = e_2\}.$$

So, we showed $e_1 \in \ker \phi$. We will show that $\ker \phi$ is a subgroup of G.

Let $\phi(G)$ be the *image* of ϕ, the set of elements of H that elements of G are mapped to. Formally,

$$\phi(G) = \{\phi(g) : g \in G\}.$$

It's true, but we'll leave it as an exercise to show, that $\phi(G)$ is a group (and so a subgroup of H).

1. Let G and H be arbitrary groups with ϕ defined so that every element of G is mapped to e_2. That is, for every $g \in G$, $\phi(g) = e_2$. Show ϕ is a homomorphism.

2. Find $\ker \phi$.

3. Let G be an arbitrary group. Let ϕ be the *identity isomorphism* that maps every element of G to itself. So, $\phi : G \to G$ and, for every $g \in G$, $\phi(g) = g$. Find $\ker \phi$.

1^{st} **Isomorphism Theorem.** If G is an abelian group, H is any group, and $\phi : G \to H$ is a homomorphism, then $G/\ker\phi \cong \phi(G)$. (If ϕ is onto—so $\phi(G) = H$—then $G/\ker\phi \cong H$).

4. Here's the simplest and still useful application. Let G be an arbitrary group. Let ϕ be the *identity isomorphism* that maps every element of G to itself. So, $\phi : G \to G$ and, for every $g \in G$, $\phi(g) = g$. What does the 1^{st} Isomorphism Theorem say in this case?