Let \((G_1, +_1)\) and \((G_2, +_2)\) be groups.

A homomorphism from \(G_1\) to \(G_2\) if there is a function which preserves the group operation. Formally, this means there is a function \(\phi : G_1 \rightarrow G_2\) such that:

\[
\text{For any } g, g' \in G_1 \quad \phi(g +_1 g') = \phi(g) +_2 \phi(g').
\]

Importantly, this map doesn’t need to be 1-1. Some elements may have the same functional value.

Let \(e_1\) be the identity element in \(G_1\) and \(e_2\) be the identity element in \(G_2\).

We will show that for any homomorphism \(\phi : G_1 \rightarrow G_2\), we have \(\phi(e_1) = e_2\).

1. Consider \(\phi(e_1 +_1 e_1)\). Right what the homomorphism condition means here.

2. Find \(e_1 +_1 e_1\).

3. Use that to rewrite the homomorphism condition.

4. You should have an equation of the form: \(b = b +_2 b\), for some \(b \in G_2\). \(G_2\) is a group.
   So what useful thing must be true about \(b\)?

5. Use that fact to solve for \(b\).

6. What can you conclude about \(\phi(e_1)\)?