Last name _____

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LARSON—MATH 401—CLASSROOM WORKSHEET 03
Examples of Fields.

Here are the *field axioms*. Given a set F and closed binary operations for *addition* ("+") and *multiplication* ("·"), the following properties are satisfied:

1. Associative addition: $a + (b + c) = (a + b) + c$.
2. Existence of an *additive identity* 0 satisfying: $a + 0 = 0 + a = a$.
3. Existence of *additive inverses*: For every a there is a b with $a + b = b + a = 0$ (b is commonly represented "$-a$").
4. Commutative addition: $a + b = b + a$.
5. Associative multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
6. Existence of a *multiplicative identity* 1 satisfying: $a \cdot 1 = 1 \cdot a = a$.
7. Existence of *multiplicative inverses*: For every non-zero a there is a b with $a \cdot b = b \cdot a = 1$ (b is commonly represented "a^{-1}").
8. Commutative multiplication: $a \cdot b = b \cdot a$.
9. Distributive property: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Examples

The real numbers $\mathbb{R}$ are a prototype example of a field.

We checked that $Z_2 = \{0, 1\}$, with addition and multiplication defined by the *Cayley tables* below, satisfies all of the field axioms; that is, $(Z_2, +.\cdot)$ is a field.

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

1. Are the natural numbers $\mathbb{N}$ a field? Explain.

2. Are the positive integers $\mathbb{Z}^{\geq 0}$ a field? Explain.

3. Are the integers $\mathbb{Z}$ a field? Explain.

4. Are the rational numbers $\mathbb{Q}$ a field? Explain.

5. Are the complex numbers $\mathbb{C}$ a field? Explain.

6. We checked that $Z_4 = \{0, 1, 2, 3\}$, with addition and multiplication defined by the *Cayley tables* below, does *not* satisfy all of the field axioms. What field axioms are satisfied for Z_4?

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

7. Let α be a root of the polynomial $p(x) = x^2 + x + 1$. Let $F_4 = \{0, 1, \alpha, \alpha + 1\}$, with addition and multiplication defined below. Is F_4 closed with respect to addition and multiplication?

$+$	0	1	α	$\alpha+1$
0	0	1	α	$\alpha+1$
1	1	0	$\alpha+1$	α
α	α	$\alpha+1$	0	1
$\alpha+1$	$\alpha+1$	α	1	0

$\cdot$	0	1	α	$\alpha+1$
0	0	0	0	0
1	0	1	α	$\alpha+1$
α	0	α	$\alpha+1$	1
$\alpha+1$	0	$\alpha+1$	1	α

8. Is F_4 a field?