Examples of Fields.

Here are the field axioms. Given a set $F$ and closed binary operations for addition (“+”) and multiplication (“."), the following properties are satisfied:

1. Associative addition: $a + (b + c) = (a + b) + c$.
2. Existence of an additive identity $0$ satisfying: $a + 0 = 0 + a = a$.
3. Existence of additive inverses: For every $a$ there is a $b$ with $a + b = b + a = 0$ ($b$ is commonly represented “$−a$”).
5. Associative multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
6. Existence of a multiplicative identity $1$ satisfying: $a \cdot 1 = 1 \cdot a = a$.
7. Existence of multiplicative inverses: For every non-zero $a$ there is a $b$ with $a \cdot b = b \cdot a = 1$ ($b$ is commonly represented “$a^{−1}$”).
8. Commutative multiplication: $a \cdot b = b \cdot a$.
9. Distributive property: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Examples

The real numbers $\mathbb{R}$ are a prototype example of a field.

We checked that $\mathbb{Z}_2 = \{0, 1\}$, with addition and multiplication defined by the Cayley tables below, satisfies all of the field axioms; that is, $(\mathbb{Z}_2, +, \cdot)$ is a field.

$$
\begin{array}{c|c|c}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|c|c}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
$$

1. Are the natural numbers $\mathbb{N}$ a field? Explain.

2. Are the positive integers $\mathbb{Z}^{>0}$ a field? Explain.

3. Are the integers $\mathbb{Z}$ a field? Explain.
4. Are the rational numbers $\mathbb{Q}$ a field? Explain.

5. Are the complex numbers $\mathbb{C}$ a field? Explain.

6. We checked that $Z_4 = \{0, 1, 2, 3\}$, with addition and multiplication defined by the Cayley tables below, does not satisfy all of the field axioms. What field axioms are satisfied for $Z_4$?

$\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}$

$\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 \\
2 & 0 & 2 & 0 & 2 \\
3 & 0 & 3 & 2 & 3 \\
\end{array}$

7. Let $\alpha$ be a root of the polynomial $p(x) = x^2 + x + 1$. Let $F_4 = \{0, 1, \alpha, \alpha + 1\}$, with addition and multiplication defined below. Is $F_4$ closed with respect to addition and multiplication?

$\begin{array}{c|cccc}
+ & 0 & 1 & \alpha & \alpha + 1 \\
\hline
0 & 0 & 1 & \alpha & \alpha + 1 \\
1 & 1 & 0 & \alpha + 1 & \alpha \\
\alpha & \alpha & \alpha + 1 & 0 & 1 \\
\alpha + 1 & \alpha + 1 & \alpha & 1 & 0 \\
\end{array}$

$\begin{array}{c|cccc}
\cdot & 0 & 1 & \alpha & \alpha + 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & \alpha & \alpha + 1 \\
\alpha & 0 & \alpha & \alpha + 1 & 1 \\
\alpha + 1 & 0 & \alpha + 1 & 1 & \alpha \\
\end{array}$

8. Is $F_4$ a field?